Game: Non cooperative game

1) $\mathcal{N} = \{1, \dots, N\}$ set of players

2) X_i, $\forall i \in \mathcal{N}$ = set of strategies

3) $J_i(x_1, \dots, x_N)$ = cost of P_i when i^{th} players plays x_i {minimizing}

4) $(x_1, \dots, x_N)$ = strategy profile

$x_{-i} = (x_1, x_{i-1}, x_{i+1}, x_N)$

$(x_i, z_{-i}) = (z_1 \dots z_{i-1}\ x_i\ z_{i+1} \dots z_N)$

P_1 \ P_2	silent	testify
silent	(1,1)	(3,0)
testify	(0,3)	(2,2)* NE

$J_1(\text{silent}, \text{testify}) = 3$

$J_2(\text{silent}, \text{testify}) = 0$

Nash Equilibrium: $(x_1^*, \dots x_N^*)$ is a Nash eq. if

$$J_i(x_1^*, \dots, x_N^*) \le J_i(x_i, x_{-i}^*) \quad \forall x_i \in X_i \quad \forall i \in N$$

	deer	rabbit
deer	(2,2)* NE	(0,1)
rabbit	(1,0)	(1/2, 1/2)* NE

{rewards} ← Outcome depends on communication

	L	M (2)	R
U	4,3	3,1	6,2
(1) M	2,1	3,4	3,6
D	3,0	9,6	2,8

maximize

can be eliminated

(1) Players are rational

	L	R
U	4,3	6,2
M	2,1	3,6
D	3,0	2,8

Player 1 has eliminated M, but player 2 doesnt know if player 1 is rational & hence cannot eliminate directly

(2) Players know that other players are rational.

(3) Each player knows that each player knows that each player is rational

Common Knowledge: A fact is said to be common knowledge amongst N if for any $i_1, \dots, i_k \in N$, $k \in N$, it holds that P_{i_1} knows that P_{i_2} knows that ... P_{i_k} knows the fact.

Game structure being C.K.

Dominance

A strategy x_i for P_i is said to be strictly dominated by $\hat{x}_i$ if:

$$J_i(x_i, x_{-i}) > J_i(\hat{x}_i, x_{-i}) \quad \forall x_{-i}$$

$\Rightarrow$ x_i will not be in any Nash Equilibrium

x_i is said to be weakly dominated by $\hat{x}_i$ if $J(\hat{x}_i, x_{-i}) \leq J_i(x_i, x_{-i})$ $\forall x_{-i}$ with strict for atleast one x_i

$$G = (N, \{X_i\}, J_i)$$

$$\hat{X}_i = X_i \setminus \{x_i\}$$

$$\hat{G} = \{N, X_{j, j \neq i}, \hat{X}_i, J_i\}$$

Set of Nash Equilibrium remains the same due to strong dominance

→ Removing strictly dominated, doesnot change N.E. set.

Suppose x_i is weakly dominated by $\hat{x}_i$

set of NE of $\hat{G} \subseteq$ set of NE of G

→ Removing weakly, makes a subset. (Doesnot add new, might remove some)

NE of $\hat{G}$

$$J_i(\underbrace{x_1^*, \dots, x_i^*}_{\neq x_i}, \dots x_N^*) \leq J_i(x_i^*, x_{-i}^*)$$

If a weakly dominated is removed, no new eq is created.

Cor: If elimination of strictly dominated strategies leads to single strategy profile $(x_1^*, \dots, x_n^*)$ then it is the only NE of original game.

Cor: Strictly dominated strategies, order doesnt matter.

	L	C (2)	R
T	1,2	2,3	0,3
(1) M	2,2	2,1	3,1
L	2,1	0,0	1,0

rewards

C : T : B : L

T : R : B : C → (2,2)

B, L, C, T → 3,2

T, C, R, L/R → (2,2) (3,2)

Order matters when weakly dominated

Auman model of incomplete information

J & P

3 racers P, R, G

J cant distinguish between R & G

J is color blind is C.K.

P maybe be CB or not.

(P is CB, winning is R)

$N = \{1, \dots, N\}$ = set of players

Ω = states of the world

$\mathcal{F}_i$ = a partition of Ω ← denotes the state distinguishibility of the players.

disjoint & union is complete

$i \in N$ ← For each player

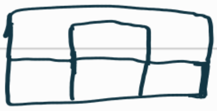
$\mathcal{F}_i(\omega)$: element of $\mathcal{F}_i$ that contain ω

Player cant distinguish between ω & $\omega' \in \mathcal{F}_i(\omega)$

(upper approximation as per observibility of i)

$\omega' \in \mathcal{F}_i(\omega) \Rightarrow \omega \in \mathcal{F}_i(\omega')$

States Ω : $\left\{ \begin{matrix} \omega_{CB,R}, & \omega_{CB,P}, & \omega_{CB,G}, \\ \omega_{NCB,R}, & \omega_{NCB,P}, & \omega_{NCB,G} \end{matrix} \right\}$

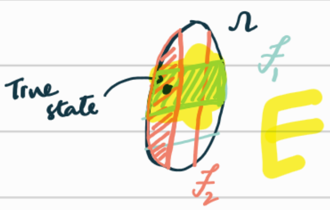
$\mathcal{F}_P$ =  $\left\{ \begin{matrix} \{\omega_{CB,R}, \omega_{CB,G}\}, & \{\omega_{CB,P}\} \\ \{\omega_{NCB,R}\}, \{\omega_{NCB,P}\}, & \{\omega_{NCB,G}\} \end{matrix} \right\}$

$\mathcal{F}_J$ =  $\left\{ \begin{matrix} \{\omega_{CB,P}, \omega_{NCB,P}\} \\ \{\text{remaining 4 states}\} \end{matrix} \right\}$

An event $E \subseteq \Omega$

Fix $\omega^* \in \Omega$

P_i knows E in ω^* if $\mathcal{F}_i(\omega^*) \leq E$



← $\mathcal{F}_1$ knows E has occured. ≈ E "obtains" in ω^*

$\mathcal{F}_2$ doesnot.

KNOWLEDGE OPERATOR → $K_i(E) = \{\omega \mid \mathcal{F}_i(\omega) \leq E\}$ All states in which i knows E

↳ Largest lower "discretization" under $\mathcal{F}$

$K_i(E)$ is the event that i knows E.

$K_j(K_i(E))$ ← j knows that i knows E

- $K_i(A \cap B) = K_i(A) \cap K_i(B)$
- $K_i(\Omega) = \Omega$
- $K_i(A) \subseteq A$
- $K_i(K_i(A)) = K_i(A)$ ← "self aware"

Knowing an event doesn't change its occurance ~ Newtonian model

Donald Rumsfield : $\{KK, UK, KU, UU\}$
$\{$ known, unknown $\}$

Event that P_i doesnot know A

- $K_i(A)^c = K_i(K_i(A^c))$
- $A \subseteq B \Rightarrow K_i(A) \subseteq K_i(B)$

Def: Let E be an event & $\omega^* \in \Omega$, then event E is C.K. in ω^* if for all $i_1, \dots, i_n \in N$, $n \in \mathbb{N}$

$$\omega^* \in K_{i_1}(K_{i_2} \cdots K_{i_n}(E))$$

Ω is always C.K.

$\emptyset$ is also C.K.

Graph $(\Omega, \mathcal{E})$

$(\omega, \omega') \in \mathcal{E}$, if $\exists\, i$ such that $\omega' \in \mathcal{F}_i(\omega)$

↳ Edge exists if any of the players cant distinguish

↳ No edge ⇒ Every player can distinguish betn them.

$C(\omega)$ ← denotes the connected component of $(\Omega, \mathcal{E})$ containing ω.

Theorem: "A is C.K. in ω" $\iff$ $C(\omega) \subseteq A$

After doing "K_i^{-1}" we cannot go outside $C(\omega)$, hence CK

Aumann model with beliefs:

Bayesian rationality

$\mathbb{P}$ = Prior, common

$\mathbb{P}(\cdot \mid F_i(\omega^*))$ = Posterior

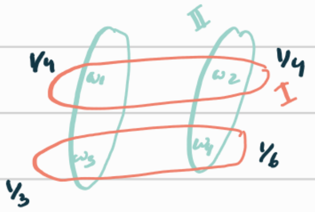
$\mathcal{E} \to$ Event

$P(\mathcal{E}) \to$ Prior

$P(\mathcal{E} \mid F_i(\omega^*)) \to$ Posterior

$\mathcal{E}' = \{\omega \mid P(\mathcal{E} \mid F_i(\omega)) = P_i\} \to$ set of states where player i gives belief P_i to event $\mathcal{E}$.

$P(\mathcal{E}' \mid P_j(\omega^*))$ = belief that player j thinks that player i has P_i for event $\mathcal{E}$



$$\mathbb{P}(\omega \mid F_I(\omega^*)) = \begin{cases} 1/2 & \omega = \omega_1 \\ 1/2 & \omega = \omega_2 \\ 0 & \omega_3 \\ 0 & \omega_4 \end{cases}$$

$$P(\omega \mid F_{II}(\omega^*)) = \begin{cases} 3/7 & \omega_1 \\ 0 & \omega_2 \\ 4/7 & \omega_3 \\ 0 & \omega_4 \end{cases}$$

$E = \{\omega \mid P(\omega_1 \mid F_I(\omega)) = 1/2\} = \{\omega_1, \omega_2\}$

↑ Becomes a subset of states

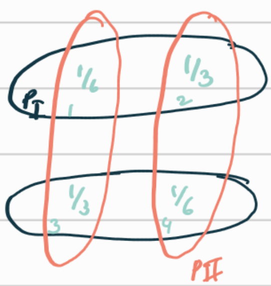
$P(E \mid F_{II}(\omega_1)) = 3/7$

↳ belief that P_2 has that P_1 has assigned belief $1/2$ for ω

$\to E = \{\omega \mid P(\omega_2 \mid F_I(\omega)) = 1/2\}$

$= \{\omega_1, \omega_2\}$

$P(E \mid F_{II}(\omega_1)) = 3/7$



$A = \{\omega_2, \omega_3\}$

$P(A \mid F_I(\omega^*)) = 2/3 = P(A \mid F_{II}(\omega^*))$

$E_1 \to \{\omega \mid \mathbb{P}(A \mid F_I(\omega)) = \overset{q_1}{2/3}\} = \Omega$

$E_2 \to \{\omega \mid \mathbb{P}(A \mid F_{II}(\omega)) = \overset{q_{II}}{2/3}\} = \Omega$

Events E_1 & E_2 is common knowledge {in every $\omega^* \in \Omega$}

Aumann's agreement theorem

Thm: Let $A \subseteq \Omega$ be an event.
Let $q_I, q_{II} \in [0,1]$. If the events that
E_I: "P_1 ascribes prob. q_I to A"
& E_{II}: "P_2 ascribes prob. q_{II} to A" is C.K. in ω then

$$q_I = q_{II}$$

$\Downarrow$

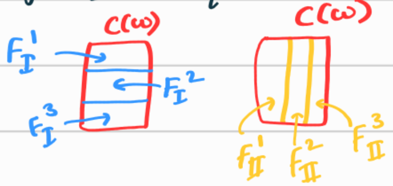
Basically: $\mathbb{P}(A \mid F_I^j) = q_I$
& $\mathbb{P}(A \mid F_{II}^j) = q_{II}$

$\left\{ q_I = \frac{P(A \cap C(\omega))}{P(C(\omega))} \right.$

$\omega \to C(\omega)$

Proof: If E is C.K. in $\omega \iff C(\omega) \subseteq E$

$\exists F_I^i, F_{II}^i$
$\in \mathcal{F}_I \quad \in \mathcal{F}_{II}$ s.t. $C(\omega) = \bigcup_j F_I^j = \bigcup_i F_{II}^i$



$\left\{ \begin{matrix} C(\omega) \subseteq E_I \\ C(\omega) \subseteq E_{II} \end{matrix} \right\}$ $\quad \mathbb{P}(A \mid F_I^j) = q_I \quad \forall j$

{From the fact that it is C.K.}

$$\frac{\mathbb{P}(A \cap F_I^j)}{\mathbb{P}(F_I^j)} = q_I$$

$$\Rightarrow \mathbb{P}(A \cap F_I^j) = q_I \, \mathbb{P}(F_I^j)$$

$$\Rightarrow \sum_j \mathbb{P}(A \cap F_I^j) = q_I \sum_j \mathbb{P}(F_I^j)$$

$$\Rightarrow \mathbb{P}(A \cap C(\omega)) = q_I \, \mathbb{P}(C(\omega))$$

$$\Rightarrow q_I = \frac{\mathbb{P}(A \cap C(\omega))}{\mathbb{P}(C(\omega))}$$

Strategy : Policy

	Opponent: Normandy x	Calais $1-x$
German y		
$1-y$		

X_i = Pure strategies, $|X_i| = m_i$

$y_i \in \Delta(X_i)$ & $y_i \in [0,1]^{m_i}$ & $\left(\sum_{k=1}^{m_i} y_k^i = 1 \right)$

$\Delta(\cdot)$ = set of prob. distributions on $(\cdot)$

mixed strategies

Both are trying to maximize

Probability of survival of S

		S	M
Shen	C	$(0,0)$	$(0.5, 0.5)$
	D	$(1,-1)$	$(0,0)$
		C	D

zero sum game : $J^1(x_1, x_2) = -J^2(x_1, x_2)$

Suppose (x_1^*, x_2^*) is N.E.

$$J^1(x_1^*, x_2^*) \le J^1(x_1, x_2^*) \quad \forall x_1$$

$$\underset{(-)J^1}{J^2}(x_1^*, x_2^*) \le \underset{(-)J^1}{J^2}(x_1^*, x_2) \quad \forall x_2$$

$$\Rightarrow \quad J^1(x_1^*, x_2) \le J^1(x_1^*, x_2^*) \le J^1(x_1, x_2^*) \quad \forall x_1, x_2$$

ZSG: NE $\Longleftrightarrow$ saddle

saddle point of the function J^1

worst case actions

$$\begin{cases} \min_{x_1} \{ \max_{x_2} J^1(x_1, x_2) \} \\ \quad \text{IV} \\ \max_{x_2} \left(\min_{x_1} J^1(x_1, x_2) \right) \end{cases}$$

$$\begin{array}{c} \\ \\ \\ \\ \\ \max\min \end{array} \begin{array}{c} \\ \left[\begin{array}{ccc} 0.5 & 2 & 1 \\ 2 & 1 & 2.5 \\ 3 & 1 & 0 \end{array}\right] \\ \underline{3} \quad \underline{1} \quad \underline{0} \\ 0.5 \quad 1 \quad 0 \end{array} \begin{array}{c} \min\ \max \\ 2 \leftarrow \\ 2.5 \\ \underline{3} \\ 4 \end{array}$$

(1 → trying to minimize
2 → trying to maximize)

x_1^*, x_2^* satisfy $\longleftarrow$ security strategies

$$\max_{x_2} J^1(x_1^*, x_2) \le \max_{x_2} J^1(x_1, x_2) \quad \forall x_1$$

$$\min_{x_1} J^1(x_1, x_2^*) \ge \max_{x_1} J^1(x_1, x_2) \quad \forall x_2$$

guaranteed payoffs with the given actions.

$$\max_y f(x,y) \ge f(x, y') \ge \min_x f(x, y') \quad \forall (x, y')$$

$$h(x) \ge g(y')$$

$$\min h(x) \ge \max g(y')$$

$$\min_x \max_y \ge \max_y \min_x$$

security strategy

worst possible min "reward" — worst possible max "lost"

$$\underline{V} \le \overline{V}$$

Thm: $\overline{V} = \underline{V}$ iff $\exists$ a saddle point

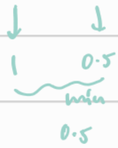
moreover, if $\exists$ saddle $\Rightarrow$ then it is composed of sec. strategy & any pair of security str is a saddle point

Every N.E. has same value

S

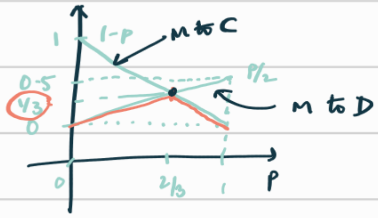
		C (q)	D ($1-q$) min	max
max	C (p)	0	0.5	→ 0
	D ($1-p$)	1	0	→ 0

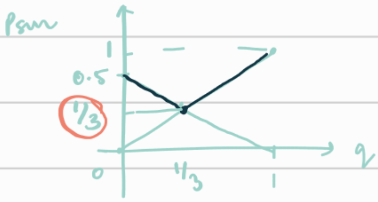
} 0



$2/3 \to C$

$1/3 \to D$





$+ \ 0.5 \ p(1-q)$

$+ \ (1-p)\, q$

$q - 1.5pq + 0.5p$

Theorem:

Von Neumann's min max thm

For any Z.S. 2s game with finitely many pure strategies there is a saddle pt in mixed strategies

Mixed strategies

X_i = set of pure strategies

$\Delta X_i \leftarrow$ set of probability distributions over X_i

$|X_i| = m_i \quad , \quad \Delta(X_i) = \{ y^i \in \mathbb{R}^{m_i} \mid 1^T y^i = 1 \ , y^i \geq 0 \}$

$\uparrow$ set of mixed strategies

$$J^i(y^i, y^{-i}) = \mathbb{E}\left(J^i(x^i, x^{-i}) \right)$$

$$= \sum J^i(x^i, x^{-i}) \cdot y^1_{x_1} \cdots y^N_{x_N}$$

Non-cooperative games

$\Downarrow$

Independent probabilities

Nash Eq^m is mixed strategy

$(y^{1*} \ \cdots \ y^{N*})$ s.t.

$$J^i(y^{i*}, y^{-i*}) \leq J^i(y^i, y^{-i*})$$

$\forall \ y^i \in \Delta(X_i)$

$\forall \ i \in \mathcal{N}$

Mixed Strategy

zero sum game

mixed security strategy

$$\left\{ \begin{array}{ll} y^{1*} & \max\limits_{y^2} J^1(y^{1*}, y^2) \leq \max\limits_{y^2} J^1(y^1, y^2) \quad \forall y^1 \\ y^{2*} & \min\limits_{y^1} J^1(y^1, y^{2*}) \geq \min\limits_{y^1} J^1(y^1, y^2) \quad \forall y^2 \end{array} \right.$$

$$\overline{V}_m = \min_{y^1} \max_{y^2} J^1(y^1, y^2) \qquad \overline{V}_m \geq \underline{V}_m$$

$$\underline{V}_m = \max_{y^2} \min_{y^1} J^1(y^1, y^2)$$

Saddle point: $J^1(y^{1*}, y^2) \leq J(y^{1*}, y^{2*}) \leq J^1(y^1, y^{2*}) \quad \forall y^1, y^2$

Proving: $\overline{V}_m \leq \overline{V}$

$$\hookrightarrow \overline{V}_m = \min_{y^1 \in \Delta(X_1)} \max_{y^2} \sum_{x_1, x_2} J^1(x^1, x^2)\, y^1_{x_1} y^2_{x_2}$$

$$\leq \min_{x_1 \in X_1} \max_{y^2 \in \Delta(X_2)} \sum_{x_2} J^1(x^1, x^2)\, y^2_{x_2}$$

Inner can be replaced, but not outer!!

$$= \min_{x_1 \in X_1} \max_{x_2 \in X_2} J^1(x^1, x^2)$$

$$= \overline{V}$$

$$\overline{V}_m \leq \overline{V}$$

$$\overline{V}_m = \min \max\, y^{1T} A y^2$$

$$= \min_{y^1} \max_{x^2} [y^{1T} A]_{x^2}$$

$$\left\{ \begin{aligned} w &= \max_{x_2} [y^{1T} A]_{x_2} \\ &= \min \{ w \mid w \geq [y^{1T} A]_{x_2} \ \forall x_2 \} \end{aligned} \right\}$$

$$= \min_{y_1} \min_{w} \{ w \mid w\mathbb{1}^T \geq y^{1T} A, \ \mathbb{1} w \geq A^T y^1 \}$$

$$= \min_{y_1, w} w \quad \text{s.t. } \mathbb{1} w \geq A^T y^1, \ \mathbb{1}^T y^1 = 1, \ y^1 \geq 0$$

Taking dual: we have duality gap is 0 as it is linear program.

$$= \max_{y^2, t} t \qquad = \underline{V}_m$$

$$\mathbb{1} t \leq A y^2$$

$$\mathbb{1}^T y^2 = 1 \qquad y^2 \geq 0$$

Thm: For any zero sum game with in 2 players and finitely many strategies

(1) $\overline{V}_m = \underline{V}_m$

(2) There exists a saddle point in mixed strategies

(3) Every saddle pt comprises of security strategy.

(4) Every pair of security strategy is a saddle point.

$$\min_{y^1} \max_{y^2} \sum_{x_1, x_2} J^1(x_1, x_2)\, y^1_{x_1} y^2_{x_2}$$

$$= \min_{y^1} \max_{x^2} \sum_{x_1} J^1(x_1, x_2)\, y^1_{x_1}$$

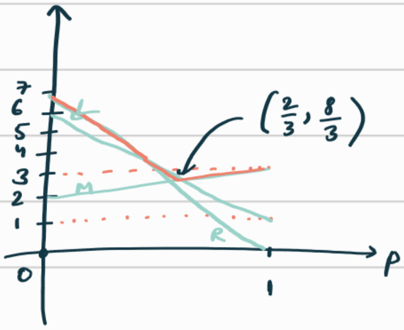
Security strategy: Maximizes the worst case gain. $\equiv$ Minimizes the worst case loss.

$$\min_{p} \max_{q_1, q_2} \mathbb{E}_{p, q_1, q_2} [\quad]$$

		L (q_1)	M (q_2)	R ($1-q_1-q_2$)
min → p	U	1	3	0
$1-p$	D	6	2	7
			↑ max	

$2 + p = 6 - 5p$

$p = 2/3$



$\rightsquigarrow \ p_R = 0$

Non Zero Sum game:

$P_i \qquad y^i \in \Delta(X_i)$

↑ mixed strategies

$$J^i(y^1 \cdots y^N) = \sum J^i(x_1 \ldots x_N)\, \underbrace{y^1_{x_1} \cdots y^N_{x_N}}_{\text{Non cooperative}}$$

cost to P_i when $(y^1 \cdots y^N$ is played$)$

$y^{1*} \cdots y^{N*}$ is a NE if

$$J^i(y^{1*}, y^{2*}, \cdots) \le J^i(y^i, y^{-i*}) \quad \forall y^i \in \Delta(x_i) \quad \forall i \in \mathcal{N}$$

Thm: Nash 1950

For any game with finitely many players and finitely many pure strategies, there exists a NE in mixed strategy.

$$R_i(y^{-i}) = \{ y^{i*} \in \Delta(x_i) \mid J^i(y^{i*}, y^{-i}) \le J^i(y^i, y^{-i}) \ \forall y^i \in \Delta(x_i) \}$$

= set of best responses of P_i to y^{-i}.

$$\arg\min_{y^1} \mathbb{E}_{x_1 \sim y^1, x_2 \sim y^2 \cdots x_N \sim y^N} [J^1(x_1, \cdots x_N)]$$

$y \in \prod_{i \in N} \Delta(X_i)$

$R(y) = R_1(y^{-1}) \times R_2(y^{-2}) \times R_3(y^{-3}) \dots \times R_N(y^{-N})$

$\subseteq \prod_{i \in N} \Delta(X_i)$

y^* is a NE iff $y^* \in R(y^*)$

y^* is a "fixed point" of R.

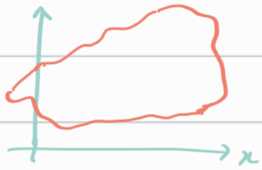
Nash Thm $\Leftrightarrow$ R has a fixed point

Kakutani fixed point thm

$T: X \rightrightarrows X$ $\qquad T: X \to 2^X$ ← set of all subsets of X

set valued map

$T(x)$



$\{(x,y) \mid y \in T(x), x \in X\}$

def: If $Gr(T)$ is closed $\Rightarrow$ $T(\cdot)$ is continuous

$g(T) = \{(x, T(x)) \mid x \in X\}$

def: $T(x)$ is convex $\forall x$ $\Rightarrow$ T is convex valued

Thm: (kakutani) Let $T: X \to 2^X$ be a set valued map s.t.

1) $X \subseteq \mathbb{R}^n$, X is convex, closed & bounded
2) T has a closed graph
3) T is convex valued

Then $\exists x^* \in X$ s.t. $x^* \in T(x^*)$

$\longrightarrow R: \prod \Delta(X_i) \to 2^{\prod \Delta(X_i)}$

$R(y)$ is convex as $R_1(y^{-1})$ is convex

Showing that it has a closed graph.

$\{y_k\}_k \in \prod(\Delta(X_i))$

$z_k \in R(y_k)$

$\{(y_k, z_k)\} \subseteq Gr(R)$

$(y^*, z^*) \in Gr(R)$

$J^1(z_k^1, y_k^{-1}) \leq J^1(y^1, y^{-1}_k)$

Fix $y^1 \in \Delta(X_1)$

taking $k \to \infty$ $\quad J^1(z^{1*}, y^{-1*}) \leq J^1(y^1, y^{-1*})$

$\forall\, y' \in \Delta(x_1)$

$z^{1*} \in R_1(y^{-1*})$

$z^* \in R(y^*) \Rightarrow Gr(R)$ is closed

$\Rightarrow$ By Kakutani $\exists$ a N.E.

Quantal Response

$\mathcal{N} = \{1 \ldots N\}$ players

X_i = set of pure strategies for P_i

$J^i : \Pi X_j \to \mathbb{R}$ cost of P_i

$J^{i,x_i}(y^{-i})$ = cost to P_i from playing pure strategies x_i when y^{-i} $= \sum_{x^{-i}} J^i(x_i, x^{-i}) \underbrace{\prod_j y^j_{x_j}}$

(Probability of profile x^{-i} being played)

$\{J^{i,x_i}(y^{-i})\}_{x_i \in X_i} \longrightarrow \Delta(X_i)$

Def: $Q_i : \mathbb{R}^{m_i} \longrightarrow \Delta(X_i)$ is a regular quantal response if it satisfies:

1) $Q_{i,x_i}(u_i) > 0 \quad \forall\, x_i \in X_i$ (interiority)

2) $Q_{i,x_i}(u_i)$ is continuous & differentiable in u_i (continuity)

Probability of taking x_i by i

3) $\dfrac{\partial Q_{i,x_i}}{\partial u_{i,x_i}} < 0$ ← (cost decreases ⇓ Increase weight)

4) $u_{i,j} < u_{i,k}$

$\Rightarrow Q_{i,j}(u_i) > Q_{i,k}(u_i)$ ← (Better action, higher probability)

Quantal Response Equilibrium

Definition: Let $Q(u) = (Q_1(u_1), \ldots, Q_N(u_N))$

A quantal response eq. is a mixed strategy profile $(y^{1*}, \cdots y^{N*})$ s.t.

$y^{i*} = Q_i(u_i(y^{-i*})) \;\forall\, i$ ← Self enforcing

$\equiv y^* = Q(u(y^*))$

Payoffs (u_i) of others profile (y^{*-i}) when into Q.R., gives the same profile (y^*)

u_i is a vector containing all payoffs. For each pure strategy.

(u_i is the pay off)

$$Q_{i,x_i}(u_i) = \left(\frac{u_{i,x_i}}{\sum_{x_i' \in X_i} u_{i,x_i'}}\right) \quad \forall\, u_{i,x_i} > 0$$

$$u_i = \begin{bmatrix} u_{i,1} \\ u_{i,2} \\ \vdots \end{bmatrix}$$

		H_q	T_{1-q}
p	H	$\mu, 0$	$0,1$
$1-p$	T	$0,1$	$1,0$

$\mu > 0$

$\mu \cdot q + (1-q)\cdot 0 = 0 \cdot q + (1-q)\cdot 1 \quad ; q = \frac{1}{1+\mu}$

counter intuitive as

$p = 1/2$

For equilibrium

For a given opponent strategy. ↑ By q.

$$Q_{1,H}(u_1) = \frac{u_{1,H}}{u_{1,H} + u_{1,T}} = \frac{\mu q}{\mu q + (1-q)} = p$$

Opponent strategy By p.

$$Q_{2,H}(u_2) = \frac{u_{2,H}}{u_{2,H} + u_{2,T}} = \frac{1-p}{1-p+p} = \frac{1-p}{} = q$$

Strategy of 1 $\boxed{p}$ → Quantal Response of 2 → Strategy of 2 $\boxed{q'}$

Quantal Responses are functions from "costs of all actions" to "mixed strategies"

τ depends on other's strategy as well

Equilibrium is the output of the function

$$Q_{1,H}(u_1) = \frac{(u_{1,H})^\lambda}{(u_{1,H})^\lambda + (u_{1,T})^\lambda}$$

$\lambda \geq 0$

Rough

$p + q = 1$

$$\frac{\mu(1-p)}{\mu(1-p)+p} = p$$

$$\mu = 2\mu p + p^2 - \mu p^2$$

$$p^2(1-\mu) + 2\mu p - \mu = 0$$

$$p = \frac{-2\mu \pm \sqrt{4\mu^2 + 4\mu(1-\mu)}}{2(1-\mu)}$$

$$\frac{-\mu \pm \sqrt{\mu}}{1-\mu}$$

$$\Rightarrow \boxed{p = \frac{\sqrt{\mu}}{1+\sqrt{\mu}}}$$

$$\frac{(\mu q)^\lambda}{(\mu q)^\lambda + (1-q)^\lambda} = p \qquad \frac{(1-p)^\lambda}{(1-p)^\lambda + p^\lambda} = q$$

$$\rightarrow \boxed{q^* = \frac{1}{1 + \mu^{\lambda^2/\lambda^2+1}} \qquad p^* = \frac{\mu^{\lambda/\lambda^2+1}}{1 + \mu^{\lambda/\lambda^2+1}}}$$

$\lambda \to \infty$

F = cumulative dist. function

Assumption : $F(x) = 1 - F(-x)$

$$F(x) = 1 - F(-x)$$

$$Q_{1,H} = F(\lambda(u_{1,H} - u_{1,T}))$$

$$Q_{i,T} = 1 - Q_{i,H}$$

$$F(x) = \frac{1}{1 + \exp(-x)}$$

		P_2	
		q H	$1-q$ T
P_1	p H	$\mu, 0$	$0, 1$
	$1-p$ T	$1, 0$	$1, 0$

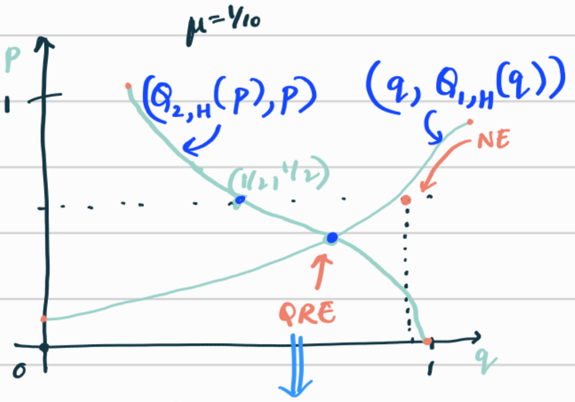
$$Q_{1,H} = F(\lambda(\mu q - (1-q))) = F(\lambda(q(1+\mu) - 1))$$

$$= \frac{1}{1 + e^{-\lambda(q(1+\mu)-1)}} = p$$

$$Q_{2,H} = F(\lambda(1-2p)) = \frac{1}{1 + e^{-\lambda(1-2p)}} = q$$

$\nwarrow Q_{2,H}(p)$

Prob row play H

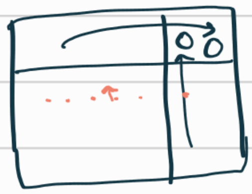


$$\begin{Bmatrix} Q_{2,H}(p) = q \\ Q_{1,H}(q) = p \end{Bmatrix}$$

$\boxed{\text{Doesnot pass through Nash Equilibrium ?}}$

- In limit will pass through. (when $\lambda \to \infty$)
- Currently due to interiority not assigning 0 to suboptimal actions

Dynamic Games



	H	T
H	1,−1	−1,1
T	−1,1	1,−1

Simultaneous Move game
↳ No information exchange
"Static games"

Homogenous production $\quad P_1 \quad P_2$
$\quad q_1 \quad q_2$

$$\text{Price} = (1 - q_1 - q_2)$$

$$\text{Cost} = cq_1,\ cq_2$$

$$\text{Profit} = (1 - q_1 - q_2)\, q_i - cq_i$$

$$J^i_{\min}(q_1, q_2) = cq_i - (1 - q_1 - q_2) q_i$$

$$-1 + c + q_2 + 2q_1 = 0 \qquad \left(\frac{1 - q_1 - c}{2}\right)$$

$q_i \in [0, \infty) = X_i$

$$-1 + c + q_1 + 2q_2 = 0$$

$$\text{NE} : \quad q_1 = q_2 = \left(\frac{1-c}{3}\right)$$

Strategy for P_1 : $[0, \infty) = \Gamma^1$

Strategy for P_2 : $\{ r^2 \mid r^2 : [0, \infty) \longrightarrow [0, \infty) \} = \Gamma^2$

↳ ✱ Space has changed

$$J^1(q_1, r^2) = cq_1 - (1 - q_1 - r^2(q_1))\, q_1$$

$$J^2(q_1, r^2) = c r^2(q_1) - (1 - q_1 - r^2(q_1))\, r^2(q_1)$$

N.E. : (q_1^{**}, r^{2**}) s.t.

$$\rightarrow J^1(q_1^{**}, r^{2**}) \le J^1(q_1, r^{2**}) \quad \forall q_1 \in \Gamma^1$$

$$\longrightarrow J^1(q_1^{**}, \gamma^{2**}) \le J^2(q_1^{**}, \gamma^2) \;\; \forall \gamma^2 \in \Gamma^2$$

$$\boxed{\begin{aligned} \gamma^{2**}(q_1) &= \frac{1-q_1-c}{2} \\ q_1^{**} &= \frac{1-c}{2} \end{aligned}}$$

← Nash Equilibrium

← First mover advantage

$$J^2(q_1^*, \gamma^{2**}) = c\left(\frac{1-c}{4}\right) - \left(1 - \left(\frac{1-c}{2}\right) - \left(\frac{1-c}{4}\right)\right)\left(\frac{1-c}{4}\right)$$

$$= \left(c - 1 + \frac{1}{2} - \frac{c}{2} + \frac{1}{4} - \frac{c}{4}\right)\left(\frac{1-c}{4}\right)$$

$$= \left(\frac{c}{4} - \frac{1}{4}\right)\left(\frac{1-c}{4}\right)$$

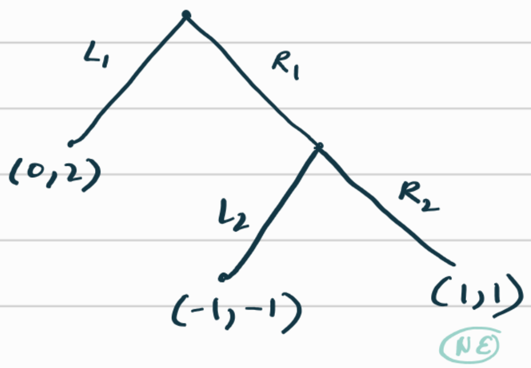
$$= \boxed{-\frac{(1-c)^2}{4^2}}$$

$$\gamma^2(q_1) = q_2^* = \frac{1-c}{3}$$

$$q_1^* = \frac{1-c}{3}$$

is also NE

{A strategy defines actions at each info set}



$$\gamma^2 : \left\{ \gamma_1^2 \begin{pmatrix} \text{do nothing at } z \\ L_2 \text{ at } y \end{pmatrix}, \; \gamma_2^2 \begin{pmatrix} \text{do nothing at } z \\ R_2 \text{ at } y \end{pmatrix} \right\}$$

$\{L_1, \gamma_1^2\}$ NE → Threat equilibrium

$\{R_1, \gamma_2^2\}$ NE

Extensive form

1) Tree, root node

2) Divide the nodes into "player sets" (partition)

when the game reaches a node in P_i's player set, P_i plays

3) Partition each player set further into information sets

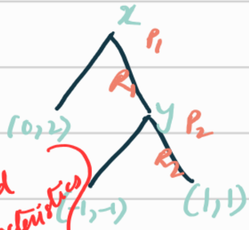
A player is unable to distinguish bet" any two nodes in the same information set.

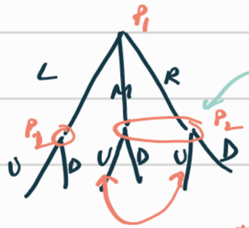
4) Costs are at the leaf nodes $(J^1, \dots, J^n)$

Difference between imperfect & incomplete information

Imperfect information: (Unaware of other's action. Non-singleton Infosets)

Incomplete information: (uninformed about characteristics of game)





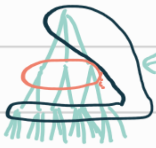
Information set

set of actions must be the same

Imperfect Information

Static game:

Each player has one information set



absent mindedness

{Not allowed as of now}

(4') Any information set cannot contain its descendants

Perfect Information: Every info set is a singleton set.

Backward Induction

or Dynamic Programming for perfect info games

If imperfect information, it fails.

Strategy: Map from info set to action.

I^i: set of info sets of P_i

for $\eta^i \in I^i$, let $x_i(\eta^i)$ = set of actions available at η^i

$$\Gamma^i = \left\{ \gamma^i \;\middle|\; \gamma^i(\eta^i) \in x_i(\eta^i) \;\; \forall \eta^i \in I^i \right\}$$

set of all "Pure" strategies

$$\gamma_1^2 = \begin{cases} U & P_1 \text{ plays } L_1 \\ D & \text{o/w} \end{cases}$$

$\begin{cases} U \\ U \end{cases}$ $\begin{cases} D \\ D \end{cases}$ $\begin{cases} D \\ U \end{cases}$

If 0 0 0 then 8 total strategies

$n = |I^i|$, m_i = actions at each info set

$m_i^{n_i}$ = Pure strategies

$$\prod_{\eta_i \in I^i} |X_i(\eta_i)| = \text{Number of Pure strategies}$$

Normal form : Matrix representation of the game with all pure strategies listed

$\Gamma^1 = \{L_1, R_1\}$

$\Gamma^2 = \{\underbrace{\{DN, L_2\}}_{\Gamma^2_1}, \underbrace{\{DN, R_2\}}_{\Gamma^2_2}\}$

	Γ^2_1	Γ^2_2
L_1	(0,2) # NE	(0,2)
R_1	(-1,-1)	(1,1) * NE

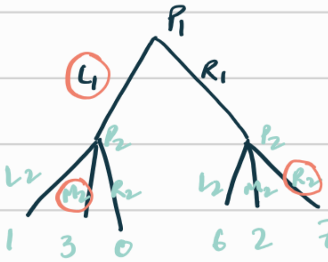
Max

To find all NE, matrix is needed.

★ Threat eq. are always weakly dominated

\# would have been missed

Threat equilibrium



1 → min

2 → max

2×9 matrix

3 is the saddle point / NE

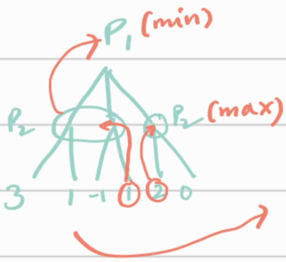
	$L_2 L_2$	$L_2 M_2$	$L_2 R_2$	$M_2 L_2$	$M_2 M_2$	$M_2 R_2$	$R_2 L_2$	$R_2 M_2$	$R_2 R_2$
L_1	1	1	1	3 #	3	3 *	0	0	0
R_1	6	2	7	6	2	7	6	2	7

{ Another equilibrium }

For zero sum games all saddle points must have the same value

Single Act game: Each player plays once.

Value of game → Final Value



	L_2	R_2
L_1	3	1
R_1	-1	1

SP

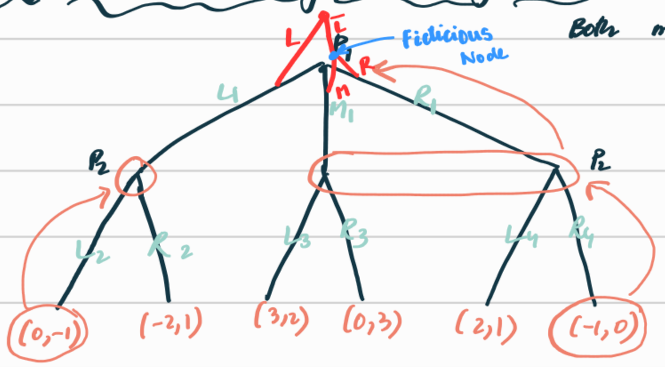
Algorithm : (1) For each info set of P_2, solve the matrix

(2) Record the saddle pt. value. If P_1 is playing 1st, then choose the lowest sp or else choose the max.

(3) Corresponding strategy is s.p. strategy

Single Act - non zero sum game

Both minimizing



LHS Game

	L_2	R_2
L_1	(0,-1)	(-2,1)

RHS Game

	L_3	R_3
M_1	(3,2)	(0,3)
R_1	(2,1)	(-1,0)

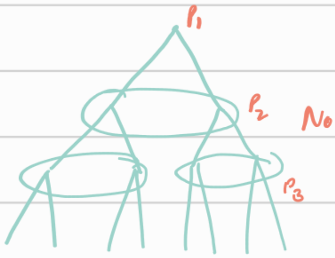
NE

$P_1 = \{L_1, M_1, R_1\}$

$P_2 = \{L_2L_3,\ L_2R_3,\ R_2L_3,\ R_2R_3\}$

	L_2L_3	R_2R_3	L_2R_3	R_2L_3
L	NE (0,-1)	(-2,1)	(0,-1)	(-2,1)
M	(3,2)	(0,3)	(0,3)	(3,2)
R	(2,1)	(-1,0)	(-1,0) NE	(2,1)

{was missed earlier}



Not decomposible into simpler games

Def^n: Nested info structure:

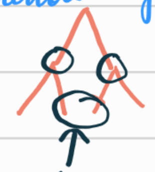
An extensive form is nested if each player has access to the information acquired by all its predecessor.

Set of players that have acted along the path leading to the info set of the player.

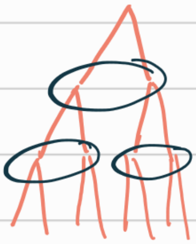
Def^n Ladder nested it is nested & only difference b/w info of P_i & P_{i-1} is the action of P_{i-1} and only at the nodes corresponding to the branches emanating from the

(?) $i+1$ cannot know actions of predecessors of i, if i didn't know.

singleton info set of P_{i-1}



Not Nested



Nested but not ladder nested

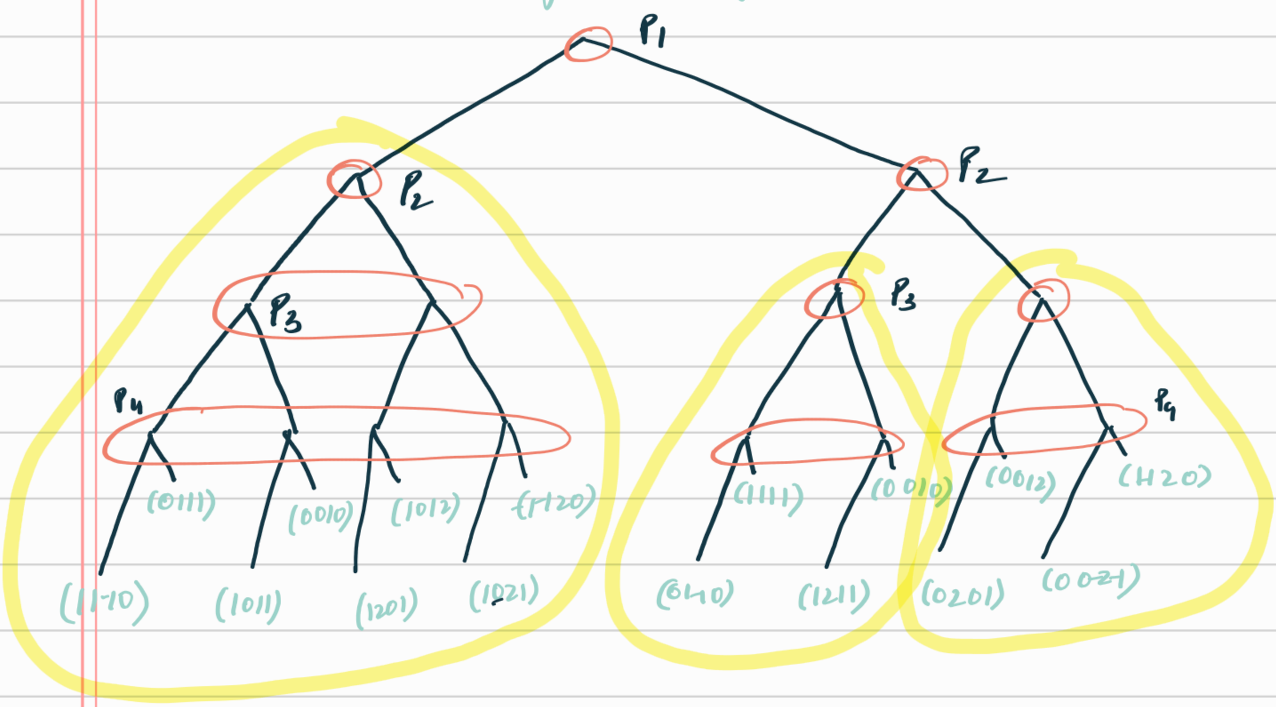
ladder nested.

Subextensive form : Take a singleton info set of P_i and subtree of all branches that enter a particular info set of P_{i+1}

Algorithm : 1) For each info set of last player, determine the subextensive form that includes all players that have same information. Solve this static game

2) Replace subextensive form with a branch of equilibrium action of the first acting player in the static subextension.

3) Repeat until you are left with the 1st acting player.



Delayed commitment type equilibrium

Single Act Game:

Each path from root to leaf intersects each player's player set at most once.

(Ask yourself why the converse does not make sense)

(Every info of inf is a super. of some info of sup ⊗)

I, II. We say I is informationally inferior to II if:

$$\forall i, \quad \forall \eta^i_{II} \in I^i_{II} \; \exists \eta^i_{I} \in I^i_{I}$$

$$\text{s.t.} \quad \eta^i_{II} \subseteq \eta^i_{I}$$

Every info set of the superior is a subset of some info set of the inferior.

Set of strategies of an inferior game is a subset of superior game.

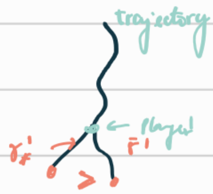
$$\forall i, \quad \Gamma^i_{I} \subseteq \Gamma^i_{II}$$

Theorem: Every NE of I (inferior) is a NE of II (superior).

Pf: Contradiction: Suppose γ^* is NE of I but not of II.

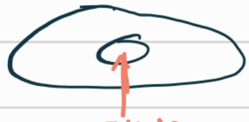
γ^* is in Γ^i_{II} & $\Rightarrow \exists$ player(1) would want to deviate from the strategy to $\tilde{\gamma} \subset \Gamma^1_{II}$

$$J^1(\tilde{\gamma}^1, \gamma^{*-1}) < J^1(\gamma^{*1}, \gamma^{*-1})$$



Let η^1 be the info set where P1 plays

$$\tilde{\gamma}^1(\eta^1_{II}) \neq \gamma^{*1}(\eta^1_{II})$$



action $\tilde{\gamma}^1(\eta^1_{II})$ could be played in η^1_{I}

Consider another strategy $\hat{\gamma}^1_{I} \in \Gamma^1_{I}$

$$\text{s.t.} \quad \hat{\gamma}^1_{I}(\eta^1_{I}) = \tilde{\gamma}^1_{II}(\eta^1_{II})$$

$$\Rightarrow J^1(\hat{\gamma}^1, \gamma^{*-1}) = J^1(\tilde{\gamma}^1, \gamma^{*-1})$$

which is not possible as γ^* is a NE.

Multi Act Games

↳ Stage wise games / feedback form

1) Tree divided into stages
2) No info set contains nodes from more than one stage
3) First acting player at each stage has singleton info set.
   Info set of other players are such that no info set contains nodes from branches from two or more different info sets of the

first acting player.

4) Each stage is a single act game.

Solution can be found in recursion type way: Feedback NE.

Project:

Finance: Review by Morris & Allen

Trading

Managerial finance

Learning from actions - Market frenzy
strategie

Security - Airport

Infrastructure

Blotto games

Resource allocation

game theoretic surveillance

intrusion detection

for fraud detection

In national parks

Green Security Games

Feedback N.E.

r_s^i = strategy of P_i in stage s.

$$\gamma^i = \underbrace{(r_1^i \ldots\ldots r_K^i)}_{K \text{ stages}}$$

$$\gamma = \underbrace{(\gamma^1, r^2, \ldots\ldots, r^N)}_{N \text{ players}}$$

Strategy profile for N players

Stage K

i^{th} player

NE

$$J^i(\underbrace{r_1^1, r_2^1, \ldots, r_{K-1}^1}_{1^{st}\text{ player}}, \underbrace{r_1^2 \ldots, r_{K-1}^2}_{2^{nd}\text{ player}}, \ldots r_{K-1}^N, \gamma_K^{1*}, r_K^{2*}, \ldots, r_K^{i*}, \ldots, r_K^{N*})$$

Should not want to switch from r_K^{i*}, irrespective of whatever strategies are played earlier by anyone

But later they are optimal

$$\leq J^i(r_1^1 \ldots r_{K-1}^1, r_1^2, \ldots r_{K-1}^2, \gamma_1^N, \ldots \gamma_{K-1}^N, \gamma_K^i, \gamma_K^{-i*})$$

$\forall\ r_K^i, \gamma_1^i, \ldots \gamma_{K-1}^1 \ldots \gamma_1^N \ldots r_{K-1}^N$
$\forall i$

No (*) for the first (K-1) stages.

Propagating these strategies upwards, we have:

But (*) for later stages.

$$J^i\left(\begin{matrix} r_1^1 \ldots\ldots r_{K-2}^1, \ldots r_{K-2}^N \\ \underline{r_{K-1}^{1*} \ldots r_{K-1}^{N*}}, r_K^{1*} \ldots\ldots r_K^{N*} \end{matrix}\right)$$

(* in K)

$$\leq J^i(r_1^1\ r_{K-2}^1 \ldots r_{K-2}^N, r_{K-1}^i, r_{K-1}^{-i*}, \gamma_K^{1*} \ldots\ldots r_K^{N*})$$

$\forall\ \gamma_{K-1}^i \quad \forall\ r_1^1 \ldots\ldots r_{K-2}^N \quad \forall i$

For the 1st stage:

$$J^i\left(r_1^{1*} \cdots r_1^{N*},\ r_2^{1*} \cdots r_k^{N*}\right)$$
$$\le J\left(r_1^i\ r_1^{-i*},\ r_2^{1*} \cdots r_k^{N*}\right)$$
$$\forall\, r_1^i \quad \forall\, i$$

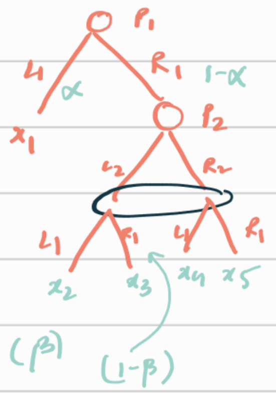
If each single act in each stage is solved for a delayed commitment eq, then the resulting eq is a delayed commitment feedback NE.

what is delayed commitment?

Memory & recall:

Prior commitment → Mixed strategy: Pick a pure strategy at random

Delayed commitment → Behavioural strategy: Choose an action at random at each info set



Set of pure strategies of $P_1 \to 4$

Set of mixed strategies of $P_1 \to$

$\Delta\Gamma^1 := \{ y \in \mathbb{R}^4 \mid y \ge 0;\ \sum y_i = 1 \}$

Set of Beh— described using $1 > \alpha, \beta > 0$

$\sigma_i \to$ Mixed strategy of P_i

$b_i \to$ Behaviour strategy of P_i

$\mu_i \to$ Either of the two

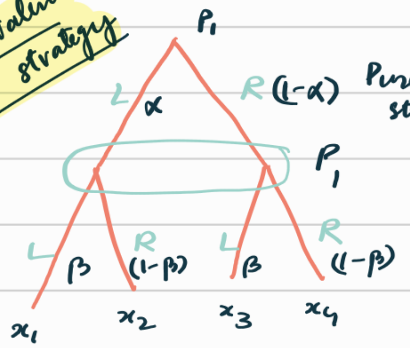
Let x be a node, $\mu = (\mu_1, \cdots, \mu_N)$

$\text{Prob}(x : \mu)$ = Probability of reaching x, under profile μ.

We say σ_i is equivalent to b_i for player i:

$$\text{Prob}(x : \sigma_i, \mu^{-i}) = \text{Prob}(x : b_i, \mu^{-i}) \quad \forall\, \mu^{-i},\ \forall\, x$$

Mixed strategy with no equivalent behaviour strategy

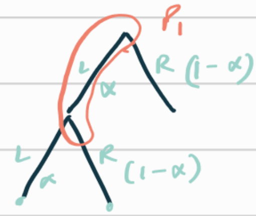


Pure strategy: LL, LR, RL, RR

$(1/2,\ 0,\ 0,\ 1/2)$ ← Mixed strategy

Behaviour strategy: $\alpha\beta,\ \alpha(1-\beta),\ (1-\alpha)\beta,\ (1-\alpha)(1-\beta)$

Behaviour strategy with no mixed strategy equivalent



$$\begin{array}{l} \text{Behaviour strategy} \\ \text{Mixed strategy} \end{array} \begin{bmatrix} \alpha^2 & \alpha(1-\alpha) & (1-\alpha) \\ p_1 & 0 & p_2 \end{bmatrix}$$

If $\sigma_i \equiv b_i \Rightarrow J^i(\sigma_i, \mu^{-i}) = J^i(b_i, \mu^{-i}) \quad \forall \mu^{-i}$

$$\sigma \equiv (\sigma_1, \dots, \sigma_N)$$
$$b \equiv (b_1, \dots, b_N)$$
$$\Rightarrow J^i(\sigma) = J^i(b) \quad \forall i$$

If for all players we have for every mixed st. $\exists$ eq beh st. & vice versa

then any NE σ^*/b^* has an equivalent b^*/σ^*.

Perfect Recall

can't forget: → Past actions

(1) → if they acted

→ what they knew

Def: P_i has perfect recall if:

(1) Each path intersects each info set of P_i ~~at least~~ at most? once

(2) If $x, \hat{x}$ are in the same info set & if x' is a predecessor of x, then there is a node $\hat{x}'$, which is a predecessor of $\hat{x}$, in the same info set as x', and the action taken at x' leading upto x must be equal to the action taken at $\hat{x}'$ leading/reaching upto $\hat{x}$.

$x' \to \hat{x}'$

$a = a$

$x \quad \hat{x}$

Kuhn's theorem

If P_i has perfect recall then every beh. strategy of P_i has an equivalent mixed strategy and vice versa.

(if only (1) holds, then beh. has an equivalent mixed)

σ_i (strategy {info set & action}) = Probability

b_i (action | infoset) = Probability

$b_i \to \sigma_i$

$$\sigma_i(r^i) = \prod_{\eta^i \in I^i} b(r^i(\eta^i) \mid \eta^i)$$

a pure strategy, giving probability of that strategy

need to check:

$$\sum_{r^i} \sigma_i(r^i) = \sum_{r^i} \prod_{\eta^i \in I^i} b_i(r^i(\eta^i) \mid \eta^i) \overset{?}{=} 1$$

$$\Gamma^i = \prod_{\eta^i \in I^i} X^i_{\eta^i}$$

Thm: If every path from root to leaf intersects an infoset of P_i atmost once, then every beh. str. of P_i has an equivalent mixed strategy.

Proof: Let $b_i \to$ behavioural strategy

$a_i(x \to \hat{x}) \longleftarrow$ The action taken by P_i at x, to steer the trajectory towards $\hat{x}$.

x: node where player i acts

$\hat{x}$: some node at future

$\to$ Nodes where P_i acts $z_1 \cdots z_\ell$

ℓ infosets

$$\text{Prob}(x; \mu_1 \cdots \mu_N) = \prod_{i=1}^{N} \text{Prob}_i(x; \mu_i)$$

$\sigma_i \equiv b_i$ if $\underbrace{\text{Prob}_i(x; b_i)}$ $= \text{Prob}_i(x; \sigma_i)$

Probability of taking actions leading upto node x by P_i

$$\text{Prob}_i(\hat{x}; b_i) = \begin{cases} \prod_{j=1}^{\ell} b_i(a_i(z_j \to \underset{z_{j+1}}{\hat{x}}) \mid n_i(z_j)) & \text{if } \ell > 0 \\ \{z_{\ell+1} = \hat{x}\} & \\ 1 & \text{if } \ell = 0 \end{cases}$$

$\Gamma^i(x) :=$ set of pure strategies of P_i in which P_i chooses actions leading to x.

$$\text{Prob}_i(\hat{x}; \sigma_i) = \begin{cases} \sum_{\gamma \in \Gamma^i(\hat{x})} \sigma_i(\gamma) & \\ 0 \; ? \text{ if doesnt act} & \end{cases}$$

$$\sigma_i(\gamma^i) = \prod_{\eta^i \in I^i} b_i(r^i(\eta^i) \mid \eta^i)$$

To prove: $\text{Prob}_i(\hat{x}, \sigma_i) = \text{Prob}_i(\hat{x}, b_i) \quad \forall \hat{x}$

$$\longrightarrow \text{LHS} = \sum_{r^i \in \Gamma^i(x)} \sigma_i(r^i) = \sum_{r^i \in \Gamma^i(x)} \prod_{\eta^i \in I^i} b_i(r^i(\eta^i) \mid \eta^i)$$

$$= \sum_{r^i \in \Gamma^i(x)} \left(\prod_{\eta^i \in I_1^i} b_i(r^i(\eta^i) \mid \eta^i) \cdot \prod_{\eta^i \in I_2^i} b_i(r^i(\eta^i) \mid \eta^i) \right)$$

Info sets on path to $\hat{x}$

Since the path intersects I_i^1 exactly once

$$\Rightarrow \quad \text{LHS} = \sum_{r^i \in \Gamma^i(x)} \left(\text{Prob}_i(x; b_i) \prod_{\eta^i \in I_2^i} b_i(r^i(\eta^i) \mid \eta^i) \right)$$

$$\sum_{r^i \in \Gamma^i(x)} \prod_{\eta^i \in I_2^i} b_i(r^i(\eta^i) \mid \eta^i) = 1$$

$$\Gamma^i = \prod_{\eta^i \in I^i} X_i(\eta_i) = \underbrace{\prod_{I_1^i}}_{\text{fixed when we have } \hat{x}} \prod_{I_2^i}$$

$$\Gamma^i(\hat{x}) = \prod_{\eta^i \in I_2^i} X_i(\eta^i)$$

{ Isomorphic Equivalence ... }

Other direction:

If P_i has perfect recall then for any x, x' in the same infoset of P_i

$$\Gamma^i(x) = \Gamma^i(x')$$

Thm: If P_i has perfect recall then every mixed has an equiv. beh.

a_i at node x

$\longrightarrow$ node reached $= x^{a_i}$

$$\Gamma^i(x^a) = \left\{ \gamma^i \in \Gamma^i(x) \;\middle|\; \gamma^i(\eta^i(x)) = a_i \right\}$$

Let σ_i be a mixed strategy

$$b_i(a_i \mid \eta^i) = \frac{\sum_{\gamma^i \in \Gamma^i(x^a)} \sigma_i(\gamma^i)}{\sum_{\gamma^i \in \Gamma^i(x)} \sigma_i(\gamma^i)} \quad x \in \eta^i$$

independent of x (in info set) due to perfect recall

$$= \frac{1}{|X^i(\eta^i)|} \quad \text{if } \sum_{\gamma^i \in \Gamma^i(x)} \sigma_i(\gamma^i) = 0$$

Now we show equivalence:

$z_1 \dots z_\ell$ nodes where P_i acts leading to $\hat{x}$.

$$\Gamma^i(z_{k+1}) = \Gamma^i\left(z_k^{a_i(z_k \to z_{k+1})}\right)$$

$$\text{Prob}_i(\hat{x} \mid b_i) = \prod_{k=1}^{\ell} b\left(a_i(z_\ell \to z_{\ell+1}) \mid \eta_i(z_k)\right)$$

$$= \prod_{k=1}^{\ell} \left(\frac{\sum_{\gamma^i \in \Gamma^i(z_{k+1})} \sigma_i(\gamma^i)}{\sum_{\gamma^i \in \Gamma^i(z_k)} \sigma_i(\gamma^i)} \right)$$

$$= \frac{\sum_{\gamma^i \in \Gamma^i(\hat{x})} \sigma_i(\gamma^i)}{\underbrace{\sum_{\gamma^i \in \Gamma^i(z_1)} \sigma_i(\gamma^i)}_{=1}} = \text{Prob}_i(\hat{x} \mid \sigma_i)$$

$\Gamma^i(z_1) \leftarrow$ all strategies.

Incomplete information v/s Imperfect information.

Bayesian Games (J. Harsanyi Game)

$N = \{1, \dots, N\}$

T_i = set of types of P_i

$T = \prod_{i \in N} T_i$

$p \in \Delta(T)$

↑ Known & common knowledge

$t = (t_1, \dots, t_N)$

P_i knows t_i

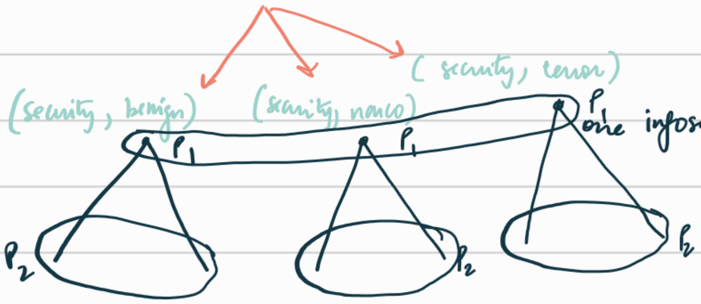
Example: $N = 2$

P_1 = Security $\quad$ $T_1 = \{\text{one type}\}$

P_2 = Passenger $\quad$ $T_2 = \{\text{benign, narco, terror}\}$
$\quad\quad$ B $\quad$ N $\quad$ T

Nature has 3 choices

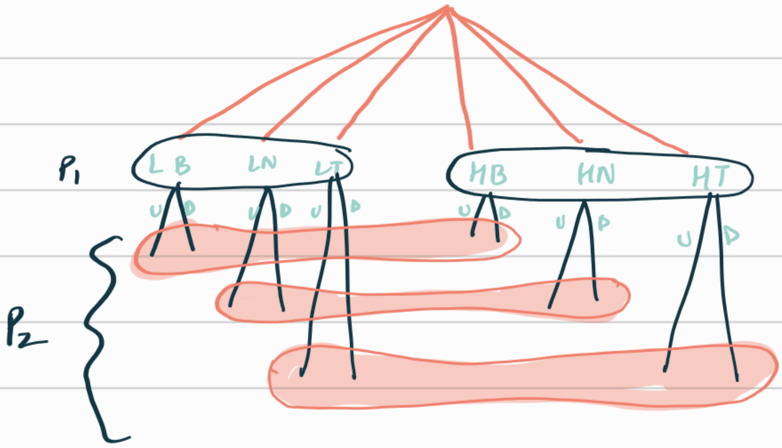
Action by Nature:



Security action not observable

$\longrightarrow T_1 = \{\text{Low, High}\}$

Nature 6 choices



If action is known then ULB & UHB will be in the infoset

(0) Nature chooses t

(1) Players know each type

(2) Players choose actions (static)

$J^i(x_i, x^{-i} | t)$ = cost of P_i when P_i plays x_i, others play x^{-i} & nature chooses t.

Player i knows $p(t^{-i} | t_i)$

$\hookrightarrow$ at stage 1: P_i can compute

$$\sum_{t^{-i}} J^i(x_i, x^{-i} / t)\, p(t^{-i} / t_i)$$

Before stage 0: P_i can compute

$$\sum_{t} J^i(x_i, x^{-i} | t)\, p(t)$$

$b_i \quad (x_i | t_i)$ ← Like an info set

$J^i \; (b_i \; , b^{-i} \,|\, t_i)$

$\{$ All $p > 0 \}$

$$= \underbrace{\sum_{t^{-i}} \sum_{x_i, x^{-i}}}_{\text{Appropiate Summation}} \underbrace{J^i(x_i, x^{-i} | t)}_{\text{Value}} \underbrace{b_i \,(a_i | t_i) \cdot b^{i}(x^{-i}, t^{-i}) \cdot p \;(t^{-i} | \, t_i)}_{\text{Probability}}$$

$$J^i(b_i, b^{-i}) = \sum_{t_i} p(t_i) \, J^i (b_i \, , \, b^{-i} \,|\, t_i)$$

$\exists \, (b_1^* \, , \dots \, b_N^*)$ NE for $J^i(b_i, b^{-i})$

$\Sigma |T_i|$ - players & Bayesian NE.

We get the NE for both the J^i's are the same.

Def NE: $\quad (b_1^* , \dots \, b_N^*)$ s.t.

$$J^i(b_i^* , \quad b^{-i*}) \leq J^i(b_i \, , \, b^{-i*}) \qquad \forall \, b_i \qquad \forall \, i$$

Bayesian NE: $\quad (b_1^* , \dots \, b_N^*)$ s.t.

$$J^i(b_i^* , \, b^{-i*} \,|\, t_i) \leq J^i \, (b_i \, , b^{-i*} \,|\, t_i) \qquad \forall \, b_i \quad \forall \, t_i \qquad \forall i$$

Theorem : BNE $\Longleftrightarrow$ NE

Let $b^* = (b_1^* , \dots , b_N^*)$ be a BNE

Let b_i be

$$J^i(b_i, b^{-i*}) = \sum p \, (t_i) \; J^i \, (b_i \, , \, b^{-i*} \,|\, t_i)$$
$$\geq \sum p \, (t_i) \; J^i \, (b_i^* , \, b^{-i*} \,|\, t_i)$$
$$= J^i \, (b_i^* \, , \, b^{-i*})$$

$\Rightarrow b^*$ is a NE $\qquad\qquad$ BNE $\Rightarrow$ NE

Let b^* be a NE, not a BNE

$\Rightarrow \exists \, t_i \, , x_i \quad J^i(x_i \; ; \, b^{-i*} \,|\, t_i) < J^i \, (b^* \,|\, t_i)$
$i \in N$

There will be atleast one pure strategy.

Def : $\hat{b}_i \, (\cdot \,|\, t_i) = \begin{cases} b_i^*(\cdot \,|\, t_i) & \forall \, t_i' \neq t_i \\ x_i \quad \text{w.p. } 1 & t_i' = t_i \end{cases}$

$$J^i(\hat{b}_i, b^{-i*}) = \sum_{t_i'} p(t_i') \cdot J^i(\hat{b}_i, b^{-i*} \mid t_i')$$

$$= \sum_{t_i' \neq t_i} p(t_i')\, J^*(b_i^*, b^{-i*} \mid t_i') + J^i(x_i; b^{-i*} \mid t_i) \cdot p(t_i)$$

$$< J^i(b^*) \longrightarrow \text{contradiction}$$

Example: **Finding Bayesian Nash Equilibrium**

I, II $\qquad T_{\mathrm{I}} = \{ I_1, I_2 \}$

$T_{\mathrm{II}} = \{ \mathrm{II} \}$

$p = (I_1, \mathrm{II}) = p(I_2, \mathrm{II}) = 1/2$

{Both are maximizing}

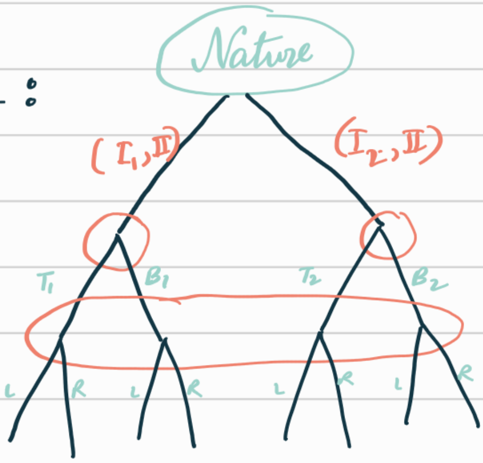
I (I_1), player II:

	L	R	
T_1	(1,0)	(0,2)	← x
B_1	(0,3)	(1,0)	← $1-x$
	↑ q	↑ $1-q$	

I (I_2), player II:

	L	R	
T_2	(0,2)	(1,1)	← y
B_2	(1,0)	(0,2)	← $1-y$
	↑ q	↑ $1-q$	

Extensive form:



→ Take $q = 0$

↓

Take best response of P_1 given $q = 0$

↓

In this case $q = 0$ not the best response.

⇓

($q=0$ or $q=1$ are not equilibrium)

As $0 < q < 1$, the player 2 must be indifferent betⁿ L & R
(assigning the probability)

⇓

L payoff $\qquad$ R payoff

$$\frac{1}{2}\left(x \cdot 0 + 3(1-x) + y \cdot 2 + (1-y) \cdot 0\right) = \frac{1}{2}\left(2x + 0 \cdot (1-x) + y + 2(1-y)\right)$$

$$3 - 3x + y \cdot 2 = 2x + 2 - y$$

$$x = \frac{1+3y}{5} \quad \leftarrow \quad 1 = 5x - 3y$$

Try $q < 1/2$ $\quad \rightarrow \quad$ $x=0$ & $y=1$ are the best response

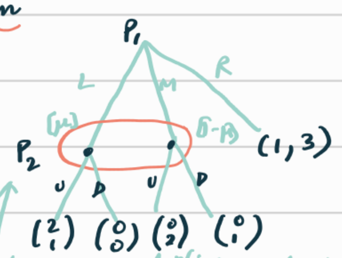
But this not satisfy $\left(\frac{1+3y}{5}\right) = x$

$q > 1/2 \longrightarrow x = 1$ & $y = 0$ are the best response

But doesn't satisfy $x = \left(\frac{1+3y}{5}\right)$

$\Rightarrow \{q = 1/2\}$ any $1 > x, y > 0$ & $x = \left(\frac{1+3y}{5}\right)$ will be the BNE

Perfect Bayesian Equilibrium



attaches a belief about probability of nodes in an infoset.

Belief: Subjective Belief

"refines the N.E."

Perfect Bayesian Eq: comprises of a system of beliefs and a profile of strategies s.t.

(i) Given their beliefs, player's strategies are sequentially rational

(ii) The belief is Bayesian-consistent with strategies

Example:

$J^2(U) = \mu + 2(1-\mu) = 2-\mu$

$J^2(D) = 0 + 1(1-\mu) = 1-\mu$

$J'(R) = 1$

$J'(L,U) = 2$

$J'(L,D) = 0$

$J'(M,U) = 0$

$J'(M,D) = 0$

$\mu = 1/2 \Rightarrow$ P2 plays U

$\Downarrow$

P1 plays L

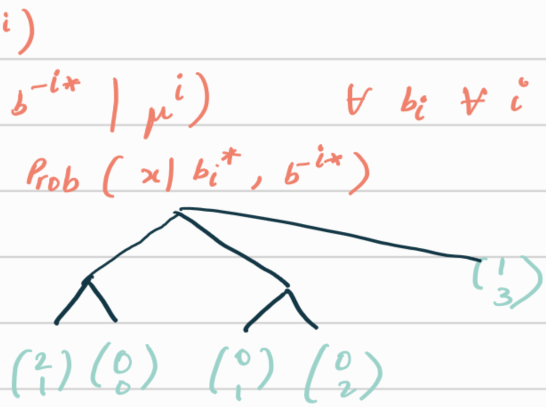
$\Downarrow$ (consistency)

$\mu = 1 \Rightarrow$ Not Equilibrium.

Definition

Minimizing: $J^i(b_i^*, b^{-i*} \mid \mu^i) \le J^i(b_i, b^{-i*} \mid \mu^i) \quad \forall b_i \; \forall i$

here $\mu^i(x) = \text{Prob}(x \mid b_i^*, b^{-i*})$



$(L \; 1/2, \; R \; 1/2)$ - P_1

μ, U_{Prob} - P2

Probability of playing u.

Payoffs : $J^2(U) = \mu + (1-\mu) = 1$

$J^2(D) = 2(1-\mu)$

$$\rightarrow \begin{cases} \mu > 1/2 \Rightarrow U \\ \mu < 1/2 \Rightarrow D \\ \mu = 1/2 \Rightarrow U \text{ or } D \end{cases}$$

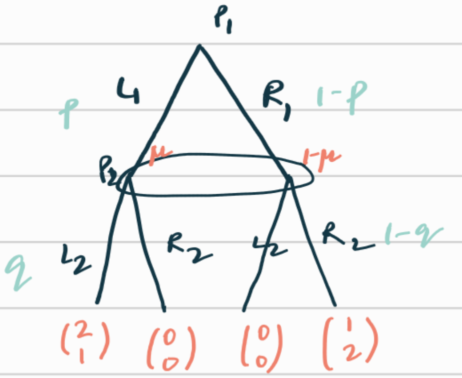
$$J^1(L/2\ \ R/2, U) = 2\times 1/2 + 1 \times 1/2 = 3/2$$

$$J^1(L/2\ \ R/2, D) = 1/2$$

$L/2\ R/2 \Rightarrow \mu = 1 \Rightarrow U \Rightarrow L$

$\{L/2\ R/2, \mu=1, U\}$ not an PBE

However $\{L, \mu = 1, U\}$ is an



$J^2(L) = \mu + 0\cdot(1-\mu) \leftarrow \mu > 2/3$

$J^2(R) = 2(1-\mu) \leftarrow \mu < 2/3$

$\boxed{\text{For } P_2}$

$\mu > 2/3 \rightarrow L$

$\mu < 2/3 \rightarrow R$

$\mu = 2/3 \rightarrow$ Either

* consistency requires P_I's belief about its cost to be the true cost.

As per its belief, it should not want to switch its strategy.

P_I's belief about its cost $\{J^1(p,q|\mu) = 1\mu q + 2(1-\mu)(1-q)$

$\{J^2(p,q|\mu) = 2pq + (1-p)(1-q)$ $\mu = 2/3$

This is the true cost of P_2 as it doesnot have a belief.

$$\frac{d}{dp}\{2pq + 1 - p - q + pq\} = 0$$

$$2q - 1 + q \Rightarrow q = 1/3$$

$$\frac{2}{3}p + (1-p)(2/3)$$

we know that $p = 2/3$ should be the equilibrium. we need to find the corresponding q.

$$\{p = 2/3,\ q = 1/3,\ \mu = 2/3\}$$

Games with Communication (static)

Simplest : randomly choose a strategy profile

Mediator → confidential communication bet^n players & a third party.

Binding agreement → Actions commited to are necessarily played.

J^i $\quad J = (J^1, \cdots, J^N)$
↑
Vector

(Maximization)

$P_1 \backslash P_2$	x_2	y_2
x_1	2 ,2	0,6
y_1	6,0	1,1

Binding Agreement : S_1

- Play (x_1, x_2) if both sign
- Play (y_1) or (y_2) if only one sign

New game:

	x_2	y_2	S_2	$\hat{S}_2$
x_1	2, 2	0,6	0,6	0,6
y_1	6,0	1,1 *	1,1	1,1
S_1	6,0	1,1	2,2 *	1,1
$\hat{S}_1$	6,0	1,1	1,1	3,3 *

S_1 weakly dominates y_1

Agreement $\hat{S}_1$:

- $\frac{1}{2}(x_1, y_2)$; $\frac{1}{2}(y_1, x_2)$ if both sign
- as before if only one player signs

Correlated Strategy

for $S \subseteq N$, correlated strategy for S is any distribution in $\Delta(X_S)$

$\mu \in \Delta(X_S)$

$$J^i(\mu) = \sum_{x_1, \cdots, x_N} \overset{\text{cost}}{J^i(x_1, \cdots, x_N)} \cdot \overset{\text{Probability}}{\mu(x_1, \cdots, x_N)}$$

$$J(\mu) = \left(J^1(\mu), J^2(\mu), \cdots, J^N(\mu)\right)$$

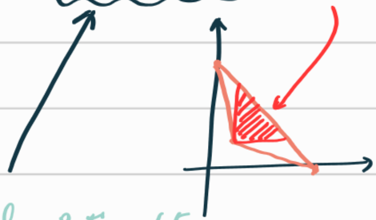
A contract $\tau = (\tau_s)_{s \subseteq N}$ where $\tau_s \in \Delta(X_s)$

$$v_i = \min_{\tau_{x^{-i}}(x^{-i})} \max_{x_i} \sum_{x^{-i}} J^i(x_i, x^{-i}) \cdot \tau_{x^{-i}}(x^{-i})$$

$$= \max_{x_i} \min_{\tau_{x^{-i}}(x^{-i})} \quad ''$$

what is the worse that can happen if I dont sign the contract? That is the security.

$$\left\{ J(\mu) \mid \underbrace{J^i(\mu) \geq v_i}_{} \quad \forall i, \; \mu \in \Delta(x) \right\}$$



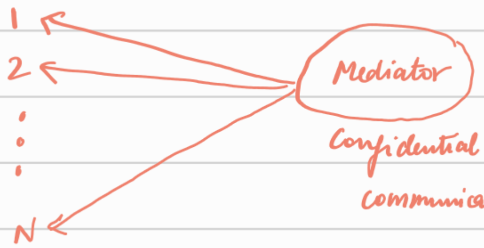
{ NE will be inside the region, but it need not be at corner }

Individual Rationality

Communication w/o binding contracts

Model communication

Mediator



Confidential One-way communication.

		P_2	
		y_1	y_2
P_1	x_1	5, 1*	0, 0
	x_2	4, 4	5, 1*

Mixed Nash Eq. : $(2.5, 2.5)$ w/o communication

(A coin is tossed, outcome is known to both)

$\hookrightarrow \frac{1}{2}(x_1, x_2), \frac{1}{2}(y_1, y_2) \longrightarrow (3,3)$

w communication, (3,3)

"Self enforcing" → Every outcome is a NE, then assuming others aren't deviating, there is no point in deviating.

with noiseless communication, players can achieve convex combination of NE payoffs.

→ Mediator takes a μ

$\mu \in \Delta(x)$

→ draws an action & tells confidentiality $x_1, x_2 \cdots x_N \sim \mu$

If i expects others to follow the recommendation ← $\sum_{x^{-i}} J^i(x_i, x^{-i}) \, \mu(x_i, x^{-i})$ ← This ought to be better

Deviation : $\delta : X_i \longrightarrow X_i$

If i deviates, but expects others to follow. ← $\sum_{x^{-i}} J^i(\delta(x_i), x^{-i}) \, \mu(x_i, x^{-i})$

This μ is known to the players.

$$\mu : \begin{cases} (x_1, x_2) & 1/3 \\ (y_1, y_2) & 1/3 \\ (y_1, x_2) & 1/3 \end{cases}$$

- If player 1 hears "x_1", he knows that 2 heard "x_2".

  1 ⇒ Assuming 2 plays "x_2" ⇒ P_1 sticks to x_1

  2

- If player 1 hears "y_1", he knows that 2 heard $\begin{cases} y_2 \to 1/2 \\ x_2 \to 1/2 \end{cases}$

  x_1 2.5, 0.5

  y_1 2.5, 4.5

  ⇒ No point in deviating

  ⇒ It is in P1's interest to stick ~~to~~ y_1.

⇒ with a mediator : $(3\frac{1}{3}, 3\frac{1}{3})$

NE (No Comms)		Direct Comms		Mediator
2.5, 2.5	$\leq$	3, 3	$\leq$	$3\frac{1}{3}, 3\frac{1}{3}$

convex combination of NE.

Corelated Equilibrium

$\mu \in \Delta(x)$ s.t.

(J is Cost) $$\sum_{x \in X} J^i(x_i, x^{-i}) \, \mu(x_i, x^{-i}) \leq \sum_{x \in X} J^i(\delta(x_i), x^{-i}) \, \mu(x_i, x^{-i}) \quad \forall \delta, \forall i \in N$$

$$\sum_{x^{-i}} J^i(x_i, x^{-i})\, \mu(x_i, x^{-i}) \le \sum_{x^{-i}} J^i(x_i', x^{-i})\, \mu(x_i, x^{-i})$$

$\forall\, x_i, x_i' \in X$

$\forall\, i \in N$

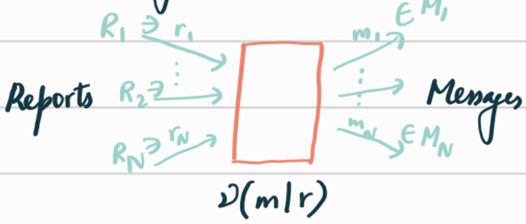
Find mediation strategy which min $\sum_{i \in N} \sum_{x} J^i(x)\, \mu(x)$

s.t. (*)

Vagueness introduced by Mediator

by inefficient communication

Communication System



$\nu(m|r)$ Prob. of generating message profile from report profile r.

Strategies set for P_i : $B_i = \{ (r_i, \delta_i) \mid r_i \in R,\ \delta_i : M_i \to X_i \}$

$$J^i(r, \delta) = \sum_{m \in M} \nu(m|r)\, J^i\big(\delta_1(m_1), \delta_2(m_2), \dots, \delta_N(m_N)\big)$$

all r_i all δ_i

Mixed strategy on B_i, $\sigma_i(r_i, \delta_i)$

$$\bar{J}^i(\sigma_i, \sigma^{-i}) = \sum_{r,\delta} J^i(r, \delta) \prod_j \sigma_j(r_j, \delta_j)$$

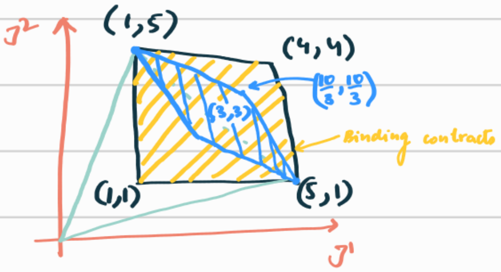
Defining Equilibrium: $\sigma_1^*, \dots, \sigma_N^*$

such a μ is a correlated Equilibrium.

This is the source of vagueness.

$$\mu(x) = \sum_{r,\delta} \sum_{m \in \delta^{-1}(x)} \prod \sigma_i(r_i, \delta_i)\, \nu(m|r)$$

		P_2	
		x_2	y_2
P_1	x_1	(5,1)	(0,0)
	y_1	(4,4)	(1,5)



$$\blacksquare = \left\{ J(\mu) \;\middle|\; \sum_{x^{-i}} J^i(x_i, x^{-i})\,\mu(x_i, x^{-i}) \geq \sum_{x^{-i}} J^i(x_i', x^{-i})\,\mu(x_i, x^{-i}) \quad \forall\, x_i, x_i',\ i \in N,\ \mu \in \Delta(X) \right\}$$

$\mathcal{G}$ = original game

$\mathcal{G}_\nu$ = game with communication system ν.

σ^* is a mixed strategy eq of $\mathcal{G}$.

σ_ν^* is a eq of $\mathcal{G}_\nu$.

$$\sigma_\nu^{i*}(\sigma_i, \delta_i) = \begin{cases} \dfrac{1}{|K_i|}\, \sigma_i^*(x_i) & \text{if } \delta_i \equiv x_i \\ 0 & \text{if } \delta_i \text{ not constant.} \end{cases}$$

"Babbling Equilibrium"

{assign non zero probabilities to only constant functions, with value as the original mixed strategy}

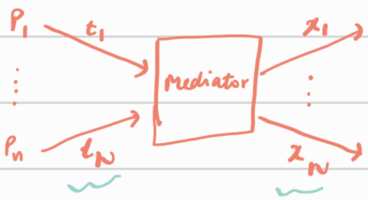
Revelation Principle

Any eq. of a game with Communication betn players can be implemented through mediation.

$$(N, \{X_i\}, \{T_i\}, p, \{J^i\})$$

$\{T_i\}$: types

p: probability with which types are chosen.



Need not be the true type or action taken

Mediation plan : $\mu(x|t) \in \Delta(X|T)$

$J^i(x,t)$ = cost of P_i from actions x, in type profile t.

$$J^i(\mu \mid t_i) = \sum_{t^{-i}\in T^{-i}} \sum_{x\in X} p(t^{-i}\mid t_i)\,\mu(x,t)\,J^i(x,t)$$

Interim payoff

when ALL players tells type correctly (truthfully) & obeys mediator, after knowing its true type.

Interim payoff after knowing his true type t_i, assuming all players reveal their true type and obey recos of the mediator

Strategies:

The language is fairly general. Cases when one doesnot want to reveal some "sensitive" info about type can be equivalently be represented by revealation principle.

$$B_i = \{(s_i, \delta_i) \mid s_i \in T_i,\ \delta_i : X_i \to X_i\}$$

$$\bar{J}^i(\{s_j, \delta_j\}_{j\in N}, t) = \sum_{x\in X} \mu(x\mid s)\, J^i(\{\delta_j(x_j)\}_{j\in N}, t)$$

others are truthful, i is lying, others are lying, i is not obedient

$$\hat{J}^i(\mu, s_i, \delta_i \mid t) = \sum_{t^{-i}\in T^{-i}} \sum_{x\in X} \mu(x\mid t^{-i}, s_i)\, J^i(\delta_i(x_i), x^{-i}, t) \cdot p(t^{-i}\mid t_i)$$

Payoff of P_i from (s_i, δ_i) assuming others obey & tell truth.

Incentive Constraints:

$$\underbrace{J^i(\mu \mid t_i) \geq \hat{J}^i(\mu, s_i, \delta_i \mid t_i)}_{\text{Incentive to tell truth \& obey}} \quad \forall (s_i, \delta_i) \in B_i$$

μ is called incentive compatible if incentive constraints hold true for each player.

Incomplete payoff game with interim payoffs: $\hat{J}^i(\mu, \delta_i, s_i | t_i)$ $\Updownarrow$ G_μ

Reformating as a Bayesian game & looking for Bayesian Nash Equilibrium

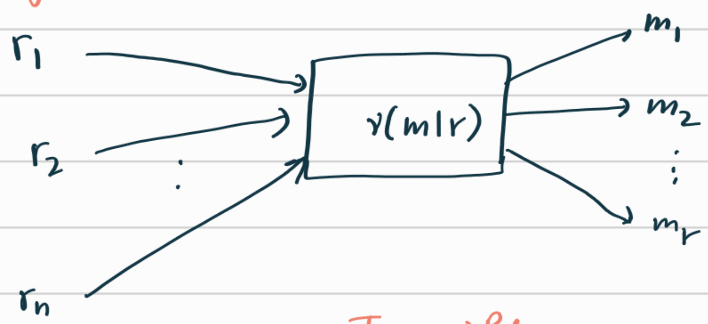
For a μ, we can find an equilibrium and then check if it is "incentive capatible"

(μ) Medition plan is incentive capatible if truthful revelation + obediance is a BNE of G_μ.

Equivalent to when:

$$J^i(\mu, \delta_i, s_i | t_i) \le J^i(\mu | t_i)$$

μ is Bayesian corelated Equilibrium.



$r_i : T_i \to R_i$ or $r_i : T_i \to R_i$

$\delta_i : M_i \times T_i \to X_i$ $\delta_i : M_i \to X_i$

strategy space: $\sigma_i(r_i, \delta_i | t_i)$

$$G = (\{J^i\}, \{X_i\}, \{T_i\}, N)$$
$$G_\nu = (\{\bar{J}^i\}, \{B_i\}, \{T_i\}, N)$$

$$\bar{J}^i(r, \delta | t_i) = \sum_{m, t^{-i}} J^i(\{\delta_j(m_j)\}_j, t)\, \nu(m|r)\, p(t^{-i}|t_i)$$

$$\bar{J}^i(\sigma | t_i) = \sum_{\substack{m, t^{-i} \\ r, \delta}} J^i(\{\delta_j(m_j)\}_j, t)\, \nu(m|r)\, p(t^{-i}|t_i) \cdot \underbrace{\prod_j \sigma_j(r_j, \delta_j)}$$

Each player with a different mixed strategy

For every ν & every BNE of G_ν $\exists \mu$ s.t.

(1) The mediation plan is incentive compatible.

Probability of taking x given types

$$\mu(x,t) = \sum_{r,\delta,m \in \delta^{-1}(x)} \prod_{j \in N} \sigma_j^*(r_j, \delta_j | t_j)\ \nu(m|r) \quad (\#)$$

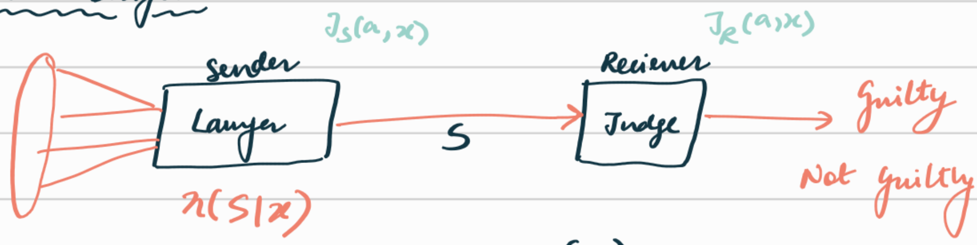
Conversely:

for every incentive compatible mediation plan
$\exists\ \nu$ s.t. $\exists$ a BNE σ^* of G_ν s.t. (#) holds.
{This is fairly straightforward}

Truthful ≡ Adverse Selection: Not knowing the type of the agent, can lead to "healthy paying for unhealthy"

Obedience ≡ Moral hazard: Complying to a signed contract. "Everyone becomes IIC agent but does no work"

Information Design



$$p(x)$$
$$\downarrow$$
$$p(x|s) = \frac{p(x|s)}{p(s)} = \frac{\pi(s|x)\,p(x)}{\sum_{x \in X} \pi(s|x)\,p(x)}$$

X { guilty, rich
non guilty, rich
guilty, not at location }

"Nudge" ← Richard Thaler (Nobel Laureate)