

can achieve
a global property
which they can't
measure

communication is limited.

Rendezvous Algorithms.

Getting the mathematical models to run on hardware is a major issue.

Come to same place & time

(not specifically at a specific location)

$\{x_i\}$ position of i^{th} agent.

$$\dot{x}_i = u_i$$

Agents can access only relative displacements. First order dynamics.

for 2 agents: $\begin{cases} \dot{x}_1 = r_1(x_1 - x_2) \\ \dot{x}_2 = r_2(x_2 - x_1) \end{cases}$

$$(\dot{x}_1 - \dot{x}_2) = -(r_1 + r_2)(x_1 - x_2)$$

$$\dot{e} = -(r_1 + r_2) e$$

4 $r_1 + r_2 > 0 \Rightarrow e \rightarrow 0$

→ Objective : meet at same location

↳ Distributed sensor Networks

↳ People with opinion

↳ constant velocity - agree on heading

state of robot $i \in \{1, \dots, N\}$, $x_i \in \mathbb{R}^d$

Assume we can control robot velocity

$$\overset{0}{x}_i = u_i$$

Global knowledge maybe too much to ask

Reasonable in relative displacement.

$$\{x_i - x_j\}$$

$$u_i = g_i(x_1 - x_i, \dots, x_N - x_i)$$

$$\left. \begin{array}{l} N=2 \\ d=1 \end{array} \right\} \quad \begin{array}{l} \dot{x}_1 = \sigma_1(x_2 - x_1) \\ \dot{x}_2 = \sigma_2(x_1 - x_2) \end{array}$$

$$e(t) = x_1(t) - x_2(t)$$

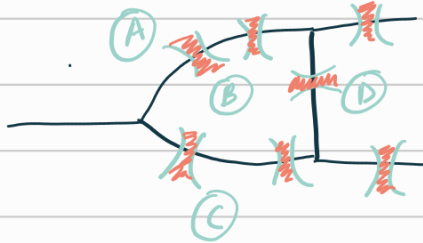
$$\dot{e}(t) = -(r_1 + r_2) e(t)$$

$$N_v = \{ u \in V \mid (u, v) \in E \}$$

neighbourhood set

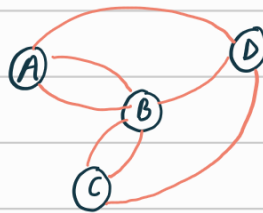
Dyadic Expansion: $A = \sum_{i=1}^n \lambda_i v_i v_i^T$

left Eigen vectors v_i right Eigen vector v_i^T



Can you go through the 7 bridges without "doubling" any?

Cannot be done!!



Hit all edges exactly once.

Euler's Prog: If not start/end $\Rightarrow$ Even # of Edges.

A $\rightarrow 3$
B $\rightarrow 5$
C $\rightarrow 3$
D $\rightarrow 3$

Not Possible

Graph is connected : 3 path between any 2 nodes

Dis connected : Not Connected

cycle : Simple path with same start and end node.

An undirected graph is acyclic if contains no cycles

Connected acyclic graph is called a tree.

undirected

Complete Graph K_N : $V = \{v_1, \dots, v_N\}$

$$E = V \times V$$

Cycle Graph C_N :

Star Graph S_N : has $N+1$ vertices



Line graph L_5 :

given $G = (V, E)$, then $G' = (V', E')$ is a subgraph if it is graph $\{ \begin{matrix} V' \subset V \\ E' \subset E \end{matrix} \}$

Subgraph is "induced by V' " if it contains any edge in E that connects

vertices in V' .

Subgraph is spanning if $V' = V$

Directed Graph: Does not hold: $(u, v) \in E \not\Rightarrow (v, u) \in E$

Possible that (u, v) & (v, u) are edges.

Self loops are allowed: (u, u)

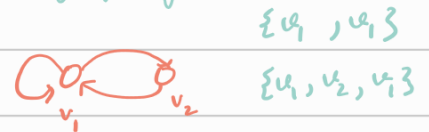
"Digraphs"

(u, v) : u is "tail" & v is "head"

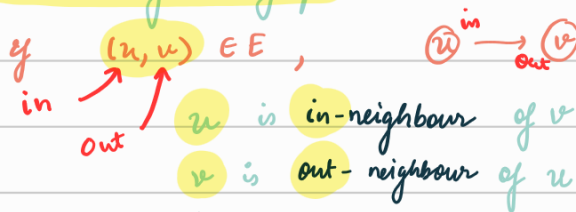
$$E_{ij} = (v_i, v_j)$$

Directed path: Sequence of nodes obeys directionality of edges.

Directed cycle: As few as 1 or 2 edges.



DAAG: Directed Acyclic Graph



$$N_v^{\text{in}} = \{u : (u, v) \in E\}$$

$$N_v^{\text{out}} = \{u : (v, u) \in E\}$$

$$\text{in-degree of } v = \|N_v^{\text{in}}\| = \text{deg}_v^{\text{in}}$$

Node with 0 in-neighbour $\leftarrow$ Source
" 0 out-neighbour $\leftarrow$ Sink

Strongly Connected: Directed Path betⁿ any 2 nodes

Weakly Connected: If "undirected version" is connected.

Balanced Graph: if Digraph has $\text{deg}^{\text{in}}(v) = \text{deg}^{\text{out}}(v)$ $\forall v \in V$
in-degree = out-degree

A leaf of a tree is a node w/degree = 1

Thm: Every tree ($n \geq 2$) has atleast 2 leaves.

Proof: By Induction (Base $n=2$) Only tree is K_2 $\circ-\circ$ has 2 leaf nodes

Suppose true for every tree with m nodes,

(we show that holds for $m+1$)

G be a some graph with $m+1$ nodes.

Suppose has no leaves then there will be a cycle.

G has at least one leaf, remove it.
 We land $G' \rightarrow$ has n nodes. G' is still a tree. Has 2 leaves.
 Adding v_0 can remove at most 1 leaf. $\Rightarrow G$ has at least 2 leaves.

Weighted graphs: Add weights to edges.

$\uparrow$ usually positive weights

Mathematically, weighted graph: $G = (V, E, \{a_e\}_{e \in E})$
 $\uparrow$ weights

Unweighted is special case if $a_e = 1$ if $e \in E$
 $a_e = 0$ if $e \notin E$

Take $V = \{v_1, \dots, v_n\}$

Weighted in-degree of v_i : $\sum_{j=1}^n a_{ji}$
 "out-degree": $\sum_{j=1}^n a_{ij}$

Isomorphic Graphs

$G = (V, E)$ $G' = (V', E')$

$\exists$ Bijective $\beta: V \rightarrow V'$ st $(v_i, v_j) \in E \Leftrightarrow (\beta(v_i), \beta(v_j)) \in E'$

$G = (V, E)$ is bipartite if: $\rightarrow V$ can be partitioned into V_1, V_2
 $(V = V_1 \cup V_2 \text{ \& } V_1 \cap V_2 = \emptyset)$

$\rightarrow$ Any edge $e \in E$ is of form
 $e = \{v_1, v_2\}$ with $v_1 \in V_1, v_2 \in V_2$

3 kinds of graphs:

- 1) Static Graphs: Fixed once & for all.
- 2) Random Graphs: Edge set changes over time, dictated by some random process
- 3) State-dependent: E depends on the state of system
 Proximity Graph

Δ -Disk Proximity Graph $x_i \in \mathbb{R}^d$ state of robot i .

It has node v_i in some graph

$V = \{v_1, \dots, v_n\}$

at time t $(v_i, v_j) \in E(t) \Leftrightarrow \|x_i(t) - x_j(t)\| \leq \Delta, i \neq j$

Edge partition: $E_1, \dots, E_n$ s.t. $\bigcup_{i=1}^n E_i = E$ & $E_i \cap E_j = \emptyset$

Thm: E can be partitioned into cycles iff every vertex has even degree.

Proof: (2f) 1) Suppose every vertex has even degree.

Find a cycle as follows:



Start @ some node, start walking.

For every new node, there is some edge to exit because degree is even.

Will eventually intersect an already visited node $\Rightarrow$ a cycle.

2) Collect edges, remove from graph, all nodes still have even degree. Repeat.

Only if: Suppose a partitioning exists, if some node is part of k cycles $\Rightarrow \deg = 2k$

Lemma $\sum_{i=1}^n \deg(v_i) = 2|E|$
of edges

Triangle: cycle of length 3 is called a triangle.

Thm: If $|E| > n^2/4$, then G must contain a triangle.

Proof: Pick edge $\{v_0, v_1\} \in E$. (Suppose G is triangle-free)

(As triangle free) $\deg(v_0) + \deg(v_1) \leq n$ (*)

$(n-2) + 2$
remaining vertices Edge twice counted

Add it for every edge

$\Rightarrow \deg^2(v_0) + \dots + \deg^2(v_n) \leq n|E|$

Triangle free
 $\Downarrow$
 $\leq n^2/4$

$$\frac{\sum \deg^2(v_i)}{n} = \frac{2E}{n} \leq \sqrt{\frac{E \deg^2}{n}}$$

$$\Rightarrow \frac{4E^2}{n^2} \leq |E| \Rightarrow |E| \leq \frac{n^2}{4}$$

We can use Cauchy Schwarz inequality: $\left(\mathbf{1}^T \begin{bmatrix} \deg(v_1) \\ \vdots \\ \deg(v_n) \end{bmatrix} \right)^2 \leq \|\mathbf{1}\|^2 \left\| \begin{bmatrix} \deg(v_1) \\ \vdots \\ \deg(v_n) \end{bmatrix} \right\|^2$

Algebraic Graph Theory

$$\dot{x}_i = \sum_{j \in N_i} (x_j - x_i) \quad (\Rightarrow \text{Graph})$$

From a graph G , we will have 4 matrices:

- 1) Adjacency Matrix
- 2) Degree Matrix
- 3) Laplacian matrix
- 4) Incidence matrix

Adjacency Matrix: A is $n \times n$

$\uparrow$ symmetric for undirected graphs

$$A_{ij} = \begin{cases} 1 & \text{if edge connects } i \text{ to } j \\ 0 & \text{o.w.} \end{cases}$$

Degree Matrix: Δ is $n \times n$

$$\Delta_{ij} = \begin{cases} \deg(v_i) & \text{if } i=j \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} \Delta_{ii} &= \sum_{j=1}^n A_{ij} \\ &= \sum_{j=1}^n A_{ji} \end{aligned}$$

In directed case:

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{else} \end{cases}$$

$\uparrow$ not symmetric

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$



$\Delta_{in} \leftarrow$ In-degree Matrix

$\Delta_{out} \leftarrow$ out-degree matrix

$$\Delta_{out} = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

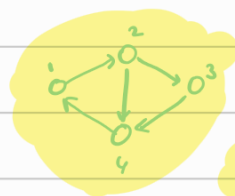
$$= \text{diag}(A \cdot \mathbb{1}_n)$$

$$\Delta_{in} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \text{diag}(\mathbb{1}_n^T \cdot A) = \text{diag}(A^T \mathbb{1}_n)$$

$$\Delta = \text{diag}(A \mathbb{1})$$

Directed graphs



$a_{ij} = 1$ if (i,j) is edge

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$A \neq A^T$

$$\Delta_{out} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \text{diag}(A \mathbb{1})$$

$$\Delta_{in} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \text{diag}(A^T \mathbb{1})$$

Weighted graphs

$\leftarrow$ Entries contain the weight

Powers of Adjacency Matrix (unweighted, undirected)

Tells us about the paths in the graph.

$$A^1 = A \rightarrow A_{ij} = 1 \text{ iff exists } \underline{\text{edge from } i \text{ to } j.}$$

length of path = 1

$$[A^2]_{ij} = (A \cdot A)_{ij} = \sum_{h=1}^n A_{ih} A_{hj}$$

= 1 iff $i \rightarrow h \rightarrow j$ is a path in graph

= # of length 2 paths from i to j

$$[A^n]_{ij} = \# \text{ of paths of length } n \text{ from } i \text{ to } j$$

Laplacian Matrix

Laplacian for unweighted undirected graphs:

$$A = A^T \text{ binary adjacency matrix}$$

$$\Delta \text{ degree matrix}$$

$$L = \Delta - A$$

- (1) $\left\{ \begin{array}{l} \rightarrow \text{Non negative on diagonal} \\ \rightarrow \text{Non positive on off-diagonal} \end{array} \right.$

- (2) L is symm as A is symm.

Rendezvous Algorithm:

$$\begin{aligned} \dot{x}_i &= \sum_{j \in N_i} (x_j - x_i) \\ &= \deg(v_i) (-x_i) + \sum_{j \in N_i} x_j \\ &= -\deg(v_i) x_i + \sum_{j=1}^n a_{ij} x_j \\ \dot{x} &= (-D + A)x = -Lx \end{aligned}$$

Laplacian Potential

$$x^T L x = \sum_{\{i,j\} \in E} a_{ij} (x_i - x_j)^2$$

Spring Network



Assume Spring constant = 1

$$\text{Force on cart } i = \sum_{j \in N_i} (x_j - x_i) = F_i$$

$$F_i = -(Lx)_i$$

$$\text{Energy in spring} : \frac{1}{2} (x_i - x_j)^2$$

$$E = \frac{1}{2} \sum_{\{i,j\} \in E} (x_i - x_j)^2 = \frac{1}{2} x^T L x$$

"All-Ones vector":

$$\mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$L\mathbf{1} = 0$$

eigenvector with eigenvalue of 0.

Three properties

(1) L is symmetric, $L = L^T$

(2) L is positive semidefinite

(3) L has real non negative eigenvalues.

$$0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$$

$$\begin{cases} P = P^T \text{ is PD if } x^T P x > 0 \quad \forall x \neq 0 \\ P = P^T \text{ is PSD } \geq 0 \quad \forall x \neq 0 \end{cases}$$

Pf: Suppose (λ, v) is eigenvalue-vector pair.

$$\langle Lv, Lv \rangle = v^* L^T L v = v^* L^2 v = v^* \lambda^2 v = \lambda^2 \langle v, v \rangle$$

$$\Rightarrow \lambda^2 = \frac{\langle Lv, Lv \rangle}{\langle v, v \rangle} \text{ is } \geq 0.$$

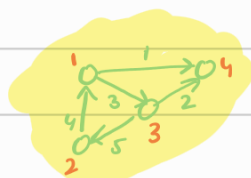
Hence, $\lambda \in \text{Real}$

$$0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$$

Became random. in $\dot{x} = -Lx$

Behaviour dictated by eigen-values of $-L$.

Incidence Matrix



(i) Orient graph with directions

$$(ii) D_{ij} = \begin{cases} 1 & \text{if } i \text{ is head of } j \\ -1 & \text{if } i \text{ is tail of } j \end{cases}$$

$$D = \begin{bmatrix} -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Claim: $D \cdot D^T = L$

$DD^T \leftarrow$ is sym, PSD & of correct size

Proof:

$$[DD^T]_{ij} \quad \longleftarrow \quad \left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right] \left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right]$$

$$[DD^T]_{ii} = [D_i \text{ row}] [D_i \text{ row}] = \sum_{i \text{ is head}} (-1)(-1) + \sum_{i \text{ is tail}} (1)(1) = \text{degree of } i$$

$$[DD^T]_{ij} = [D_i \text{ row}] [D_j \text{ row}] = \begin{cases} (+1)(-1) & \text{if } i \leftarrow j \text{ is edge} \\ (-1)(+1) & \text{if } j \leftarrow i \text{ is edge} \\ 0 & \text{otherwise} \end{cases}$$

S_3 : 4 nodes

$$\begin{aligned} \lambda_1 &= 0 & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ \lambda_2 &= 4 & \begin{bmatrix} 3 \\ -1 \\ -1 \\ -1 \end{bmatrix} \\ \lambda_3 &= 1 & \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \\ \lambda_4 &= 1 & \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \end{aligned}$$



$$= -1 \text{ if } \{i, j\} \text{ is edge}$$

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

S_n , $n+1$ nodes

$$\begin{aligned} \lambda_1 &= 0 & \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \\ \lambda_2 &= \dots & \lambda_n = -1 & \begin{bmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{bmatrix} \\ \lambda_{n+1} &= (n+1) & \begin{bmatrix} n \\ -1 \\ \vdots \\ -1 \end{bmatrix} \end{aligned}$$

Line Graph

$$\{0\} \cup \{2 - 2 \cos\left(\frac{\pi i}{n}\right); i=1, n-1\}$$

$\lambda_2 \leq 1$, gets smaller as n increases

Cycle graph w/n nodes

$$\{0\} \cup \{2 - 2 \cos\left(\frac{2\pi i}{n}\right); i=1, n-1\}$$

Intuition Behind Laplacian Matrix



$$z_i = x(ih)$$

$x(s,t)$ = temperature of beam @ position s , time t .

$$\frac{\partial x(s,t)}{\partial t} = \nabla^2 x(s,t) \quad \text{in 1D: } \nabla^2 x(s,t) = \frac{\partial^2 x(s,t)}{\partial s^2}$$



$$\nabla^2 x(s,t) \approx \frac{\frac{z_{i+1} - z_i}{h} - \frac{z_i - z_{i-1}}{h}}{h}$$

$$= \frac{z_{i+1} + z_{i-1} - 2z_i}{h^2}$$

$$= \frac{1}{h^2} \sum_{j \in \{i-1, i+1\}} z_j - z_i$$

If L is Laplacian of line graph

$$\dot{z} = -\frac{c}{h^2} L z$$

Algebraic Connectivity (Band on eigenvalues of Laplacian)

↳ Easy to compute

↳ Directly connected to behavior of algo's like Random Walks.

$$L = \Delta - A$$

$$(1) L = L^T$$

$$(2) L \succeq 0 \Leftrightarrow x^T L x \geq 0 \quad \forall x$$

$$(3) \text{ eigenvalues } 0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$$

Theorem: For an undirected graph G , the smallest eigenvalue of L is $\lambda_1 = 0$.

Moreover, $\lambda_2 > 0$ iff G is connected.

Lemma: For any matrix $M \in \mathbb{R}^{n \times m}$, $\text{null}(M) = \text{null}(M^T M)$

Proof: If $x \in \text{null}(M) \Rightarrow Mx = 0 \Rightarrow M^T M x = 0 \Rightarrow \text{null}(M) \subseteq \text{null}(M^T M)$

If $x \in \text{null}(M^T M) \Rightarrow M^T M x = 0 \Rightarrow x^T M^T M x = 0 \Rightarrow Mx = 0 \Rightarrow \text{null}(M^T M) \subseteq \text{null}(M)$

Proof of theorem: ① $\lambda_2 > 0 \Rightarrow G$ is connected

c.p. $\hookrightarrow G$ not connected $\Rightarrow \lambda_2 = 0$

$\Rightarrow L$ is block diagonal

$$L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \quad v_1 = \begin{bmatrix} \mathbf{1} \\ 0 \end{bmatrix}; \quad v_2 = \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix}$$

$\Rightarrow$ GM of $0 \geq 2$

$\Rightarrow$ AM of $0 \geq 2$.

② G is connected $\Rightarrow \lambda_2 > 0$

Strategy: Is to fully characterize null space of L , show it is 1-dim

Let $z \in \mathbb{R}^n$ be s.t. $Lz = 0$

$$Lz = DD^T z = 0 \Leftrightarrow D^T z = 0 \Leftrightarrow z^T D = 0$$

$$[z^T D]_e = z_i - z_j \text{ for } e = i \rightarrow j$$

$$\Rightarrow z_i = z_j \text{ for } \forall \{i, j\} \in E$$

Since G is connected, for any i, j nodes

there exists a path $(i, k_1, \dots, k_n, j)$ connecting i to j .

$$z_i = z_{k_1} = \dots = z_{k_n} = z_j$$

$$\Rightarrow z = \begin{bmatrix} \alpha \\ \alpha \\ \vdots \\ \alpha \end{bmatrix} = \alpha \mathbf{1} \in \text{span}(\mathbf{1})$$

$$\Rightarrow \text{null}(L) = \text{span}(\mathbf{1}) \Rightarrow \text{GM} = 1$$

$\Rightarrow$ AM = GM as L is symm.

Corollary: The multiplicity of the 0 eigenvalue in L is equal to the # of connected components in G .

Idea: Use λ_2 as a measure of connectivity, or Fiedler Eigenvalue

λ_2 is larger for "more connected" graphs.

For complete graph K_n , $\lambda_2 = n$.

Variational characterization of λ_2 :

$$\lambda_2 = \min_{\substack{\|x\|^2 = 1 \\ \text{with } x \perp \mathbf{1}}} x^T L x$$

Corollary: Monotonicity Property in λ_2

We write $G \leq G'$ if $G = (V, E)$
 $(V = V', E \subseteq E') \rightarrow G' = (V', E')$

$$\text{if } G \leq G' \Rightarrow \lambda_2 \leq \lambda'_2$$

$V' \subset V$ is a vertex cut set if removing V' from G , disconnects the graph.

$E' \subset E$ is a Edge cut set if removing E' from G , disconnects the graph.

Minimal vertex cut set is a smallest vertex cut set

Vertex connectivity: K_v : size of minimal vertex cut set

Similar for minimal edge cut set & edge connectivity.

Theorem: Let $G = (V, E)$ be undirected, unweighted graph.

$$\hookrightarrow \left(\lambda_2 \leq K_v \leq K_e \leq \min_v \deg(v) \right)$$

$$\lambda_2 \geq \frac{4}{\text{diameter}(G)}$$

This bound immediately gives: $\lambda_2 \leq 1$ for eg. line graph, star graph

Cheeger's Inequality

Given $G = (V, E)$

↑ partition V in S & S^c

$E(S) = \# \text{ of edges connecting } S \text{ & } S^c$

$$Q(S) = \frac{E(S)}{\min\{|S|, |S^c|\}}$$

$$\Phi(G) = \min_S Q(S)$$

↑ Iso perimeter number of G .

Line graph: $n \leftarrow \text{even}$

$$\Phi(G) = \frac{1}{n/2}$$

Cycle graph: $n \leftarrow \text{even}$ $\Phi(G) = \frac{2}{n/2}$

$$\deg(G) = \max_{v \in G} \deg(v)$$

$$\frac{\Phi(G)^2}{2 \deg(G)} \leq \lambda_2 \leq 2 \cdot \Phi(G)$$

Community detection via Algebraic Connectivity

separate V into 2 groups, V_1 & V_2 with max in group connectivity & min across group " .

Given V_1, V_2 assign cost as follows:

$$x \in \{-1, 1\}^n. \quad x_i = \begin{cases} 1 & \text{if } i \in V_1 \\ -1 & \text{if } i \in V_2 \end{cases}$$

$$J(V_1, V_2) = \frac{1}{4} \sum_{i,j} (x_i - x_j)^2 = \frac{1}{4} x^T L x$$

Idea: solve: $\min_{x \in \{-1, 1\}^n} x^T L x$ ①

$$\begin{aligned} x &\in \{-1, 1\}^n \\ x &\neq -\mathbb{1} \\ x &\neq \mathbb{1} \end{aligned}$$

Mixed Integer - Combinatorial in nature

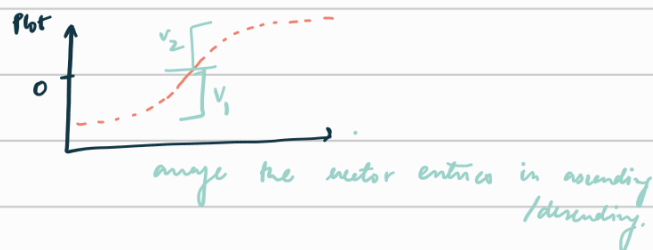
Relax the problem $\rightarrow \min_{-1 \leq x \leq 1} x^T L x$ ②

$$\begin{aligned} -1 &\leq x \leq 1 \\ x &\perp \mathbb{1} \\ \|x_{\text{avg}}\| &\leq 1 \end{aligned}$$

$$\min_{\substack{\|x\|_2 = 1 \\ x \perp \mathbb{1}}} x^T L x = \lambda_2 \quad \text{the minimizer } v_2.$$

Proposed solution: compute λ_2, v_2

$$x = \text{sign}(v_2) \quad \rightarrow \text{entry wise}$$



$$\dot{x} = -Lx \quad \text{is LTI.}$$

$L = L^T \Rightarrow$ Real Eigenvalues

$\Rightarrow$ it has a basis of orthonormal eigenvector $\{v_1, \dots, v_n\}$

$$L v_i = \lambda_i v_i, \quad \|v_i\|_2 = 1, \quad v_i^T v_j = 0 \quad \forall i \neq j.$$

Choose to write: $x(t) = \sum y_i(t) v_i$

$$\text{left multiply with } v_i^T \Rightarrow v_i^T x(t) = y_i(t)$$

$$v_i^T \dot{x}(t) = v_i^T L x(t) = \lambda_i v_i^T x(t)$$

$$\Rightarrow \dot{y}_i(t) = -\lambda_i y_i(t)$$

$$\Rightarrow x(t) = \sum_{i \in \{1, \dots, N\}} e^{-\lambda_i t} v_i v_i^T x(0)$$

Theorem Given: static, undirected, unweighted graph G

with Laplacian L , the rendezvous / consensus equation $\dot{x} = -Lx$

achieves consensus: $\lim_{t \rightarrow \infty} (x_i(t) - x_j(t)) = 0$ for all i, j

iff G is connected. Rate of convergence is λ_2 .

Where does it come to consensus?

$$\bar{x}(t) = \frac{1}{N} \sum_{i=1}^N x_i(t) = \frac{1}{N} \mathbb{1}^T x(t) = 0$$

$$\Rightarrow \lim_{t \rightarrow \infty} x_i(t) = \bar{x}(0)$$

Thm: If G is static, undirected, unweighted, connected $\dot{x} = -Lx$ drives x to $\bar{x}(0)\mathbb{1}$.

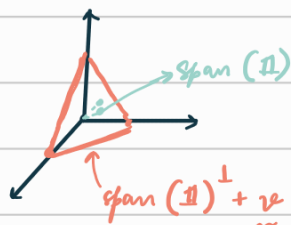
Disagreement vector:

$$s(t) = x(t) - \frac{1}{N} \mathbb{1} \mathbb{1}^T x(0) \quad \left\{ \mathbb{1}^T x(t) = \text{constant} \right\}$$

$$\left(I - \frac{1}{N} \mathbb{1} \mathbb{1}^T \right) x(0)$$

π Projection Matrix $\{\pi^2 = \pi\}$

onto disagreement subspace $(\text{span}\{\mathbb{1}\})^\perp$



$\text{span}(\mathbb{1})^\perp + v$ ← state evolves on this plane.

$$\begin{aligned} \dot{s}(t) &= \pi \dot{x}(t) = -\left(I - \frac{1}{N} \mathbb{1} \mathbb{1}^T\right) L x(t) \\ &= -L \left(I - \frac{1}{N} \mathbb{1} \mathbb{1}^T\right) x(t) = L s(t) \end{aligned}$$

$$\dot{s}(t) = -L s(t)$$

Theorem: Let G be static, undirected, ^{unweighted} connected graph

if $s = \pi x$ is disagreement vector for $\dot{x} = -Lx$ then

$$\|s(t)\| \leq e^{-\lambda_2 t} \|s(0)\|$$

Suppose $x_i \in \mathbb{R}^d$, $d > 1$

(nodal dynamics):

$$x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{id} \end{bmatrix} \in \mathbb{R}^d$$

$$z_k = \begin{bmatrix} x_{1k} \\ \vdots \\ x_{nk} \end{bmatrix} \in \mathbb{R}^N$$

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_d \end{bmatrix} \in \mathbb{R}^{Nd}$$

$$\dot{z} = \begin{bmatrix} -L z_1 \\ \vdots \\ -L z_d \end{bmatrix} = \begin{pmatrix} -L & & \\ & -L & \\ & & \ddots \\ & & & -L \end{pmatrix} z = -(\mathbb{1}_d \otimes L) z$$

(Kronecker Product)

$$A \in \mathbb{R}^{p \times q}$$

$$B \in \mathbb{R}^{r \times m}$$

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \dots & a_{pq}B \end{bmatrix} \in \mathbb{R}^{pr \times qm}$$

we look at undirected but weighted graphs.

$$A = \begin{bmatrix} 0 & w_{12} & \dots \\ w_{21} & 0 & \dots \\ \vdots & \vdots & \ddots & 0 \end{bmatrix}$$

$$\Delta = \text{degree matrix} = \text{diag}(A\mathbb{1})$$

$$L = \Delta - A$$

$$\dot{x} = -Lx$$

Let D be $n \times m$ $\leftarrow$ edges $\leftarrow$ an incidence matrix
nodes

$$W = \begin{bmatrix} w_{e_1} & & & \\ & w_{e_2} & & \\ & & \ddots & \\ 0 & & & w_{e_m} \end{bmatrix}$$

$$L = \underbrace{D \cdot W \cdot D^T}_{n \times n}$$

assumes an ordering of edges.

What is still true about L :

$$1) L = L^T$$

$$2) L\mathbb{1} = 0, \mathbb{1}^T L = 0$$

$$3) G \text{ is connected iff } \lambda_2 > 0$$

$$4) \text{Variational Characterization}$$

$$\lambda_2 = \min_{\substack{\|x\|_2=1 \\ x \perp \mathbb{1}}} x^T L x$$

Monotonicity Property: if $A \geq A'$ element-wise
then $\lambda_2 \geq \lambda'_2$

$$(Lx)_i = \sum_{j \in N_i} a_{ij} (x_i - x_j)$$

$$\text{For any } x \in \mathbb{R}^n, \quad x^T L x = \frac{1}{2} \sum_{\{i,j\} \in E} a_{ij} (x_i - x_j)^2$$

Directed Graphs

$$N_i^{\text{in}} = \{j \in V : (j, i) \in E\}$$



$$N_i^{\text{out}} = \{j \in V : (i, j) \in E\}$$

$$(j, i) \equiv i \rightarrow j \quad j \in N_i^{\text{in}}$$

$$\deg^{\text{in}}(i) = |N_i^{\text{in}}|$$

$$\deg^{\text{out}}(i) = |N_i^{\text{out}}|$$

i sends information to j .

j can access to $x_i - x_j$

Directed Consensus: $\dot{x}_i = - \sum_{j \in N_i^{\text{in}}} (x_i - x_j)$

$\dot{x}_i = x_k - x_i$

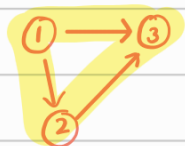
$\dot{x}_j = x_i - x_j$

$\dot{x}_k = x_j - x_k$

$L = \Delta^{\text{in}} - A^{\text{in}}$

$\dot{x} = -Lx$

$\dot{x}_i = -(Lx)_i$



$L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

$\dot{x} = -Lx = \begin{bmatrix} 0 \\ -(x_2 - x_1) \\ -[(x_3 - x_1) + (x_3 - x_2)] \end{bmatrix}$

(Everything works for weighted as well)

• Is Null (L) bigger than span(1)?

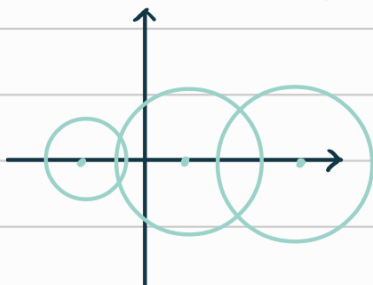
• Do any evals of L have neg. real part?

Weak connectivity is not enough.

Strong connectivity is enough

What about something in between?

Gershgorin Disk Theorem: For any $M \in \mathbb{R}^{n \times n}$, all eigenvalues are in $\bigcup_{i=1}^n \{z \in \mathbb{C} : \|z - m_{ii}\| \leq \sum_{j=1, j \neq i}^n |m_{ij}|\}$

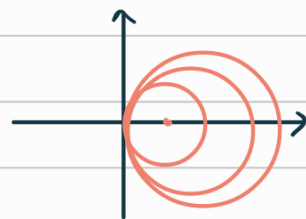


Corollary: Let L be a directed (weighted) Laplacian

All eigenvalues are in $\{z \in \mathbb{C} : \|z - \max_i \deg^{\text{in}}(i)\| \leq \max_i \deg^{\text{in}}(i)\}$

In particular, all have nonnegative real part.

Pf: The largest disk is that.

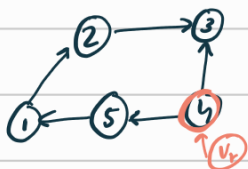


Definition: A digraph is a rooted out-branching tree (ROT) if it has some vertex v_r s.t.

for any other v , there exists a directed path from v_r to v .

• Weaker notion than strongly connected

• SC means any node can be chosen as the root.



Thm: A digraph G with n nodes contains RDT iff rank $L = n-1$.
 In this case nullity $L=1$, null $(L) = \text{span}(\mathbf{1})$

Thm: Direct consensus is achieved iff G contains a RDT as a spanning subgraph.

Invariant Notion

$$q \in \mathbb{R}^n \text{ st } q^T L = 0$$

$$\begin{aligned} \frac{d}{dt} (q^T x(t)) \\ = -q^T L x(t) = 0 \Rightarrow q^T x = \text{constant.} \end{aligned}$$

$$x(\infty) = \alpha \mathbf{1}$$

$$q^T x(0) = q^T x(\infty)$$

$$\left\{ x_i(\infty) = \frac{q^T x(0)}{q^T \mathbf{1}} \right\}$$

$$\mathbf{1}^T L = 0 \text{ iff } \deg^{\text{in}}(i) = \deg^{\text{out}}(i) \quad \forall i$$

→ Called BALANCED.

Thm: Consensus reaches avg consensus iff G is balanced & weakly connected.
 (Balanced + WC) $\Rightarrow$ RDT exists.

$$V(x) = x^T P x \quad P = P^T > 0$$

$$A^T P + P A = -Q < 0 \text{ then } A \text{ is Hurwitz.}$$

$$A \text{ is Hurwitz iff } \forall Q = Q^T > 0 \quad \exists P = P^T \text{ s.t. } A^T P + P A = -Q$$

If the pair (C, A) is observable then (?)

Consensus Algorithm

$$A \otimes B \in \mathbb{R}^{(nr) \times (mq)}$$

$$\begin{matrix} \uparrow & \uparrow \\ nxm & rxq \end{matrix} = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & \dots \end{bmatrix}$$

$$\dot{x}_i = \sum_{j \in N_i} (x_j - x_i) \rightarrow \dot{x}_{id} = \sum_{j \in N_i} (x_{jd} - x_{id})$$

"Same coordinate together"

$$X = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{N1} \\ x_{12} \\ \vdots \\ x_{Nd} \end{bmatrix}; \quad \dot{X} = \begin{bmatrix} -L x_{x1} \\ -L x_{x2} \\ \vdots \\ -L x_{xd} \end{bmatrix}$$

"Same agent together"

$$\tilde{X} = \begin{bmatrix} x_{11} \\ \vdots \\ x_{1d} \\ x_{21} \\ \vdots \\ x_{Nd} \end{bmatrix}; \quad \dot{\tilde{X}} = \begin{bmatrix} d \times Nd \\ \dots & I & \dots & -d_i I & 0 \dots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

if $j \in N_i$

$$\dot{\tilde{X}} = -(L \otimes I_d) \tilde{X}$$

$$\dot{X} = \begin{bmatrix} -L & & \\ & -L & \\ & & \ddots \\ & & & -L \end{bmatrix} X$$

$$\dot{X} = -(\mathbf{I}_d \otimes L) X \leftarrow \text{Ensemble Representation of Consensus.}$$

Different Flavours of Graph Theory

- G & $L \rightarrow$ change with time
- " " with state
- Second order or higher order systems.
- Formation Control

Proximity Graph:

$$N_i = \{j : |x_i - x_j| < \Delta\}$$

Undirected, time varying graph: $G(t) = (V(t), E(t))$

$\rightarrow$ Time varying Laplacian: $L(t)$

$$\dot{x} = -L(t)x \quad \text{LTV system.}$$

$\rightarrow$ Switches at discrete time instances.

$\rightarrow$ "Switched" or "Hybrid"

$$\mathcal{M} = \{1, \dots, M\} \text{ modes}$$

Switching signal $\sigma: \mathbb{R}^+ \rightarrow \mathcal{M}$

Switched linear system: $\dot{x}(t) = -L_{\sigma(t)} x(t)$

$$\sigma(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 2, & 1 \leq t < 2 \\ 3, & 2 \leq t < 3 \end{cases}$$

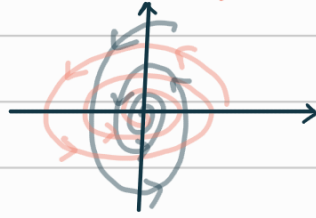


when is it stable?

- Universal stability $\leftarrow \forall$ switching signals, sys is stable

• existential stability $\leftarrow \exists$ switching signals, sys is stable

• Suppose A_1 & A_2 are Hurwitz



Using this, we can say
individual stability $\nRightarrow$ universal stability
individually instability $\nRightarrow$ universal instability

Lyapunov function : $V(x) > 0 \quad \forall x \neq 0$
 $V(0) = 0$
 $\dot{V}(x) \leq 0 \quad \leftarrow$ weak
 $< 0 \quad \forall x \neq 0 \quad \leftarrow$ strong

$V(x) = x^T P x$ $\leftarrow$ PD matrix

$$V(x) = \frac{1}{2} x^T x$$

$$\dot{V}(x) = -x^T L x \leq 0$$

$$= 0 \quad \text{if } x \in \text{Null}(L)$$

(weak condition)

$$\dot{V} = 0 \quad \text{whenever } x \in \underbrace{\text{span}\{1\}}_{\substack{\uparrow \\ \text{largest invariant set.}}}$$

Theorem: The switched linear system $\dot{x}(t) = A_{\sigma(t)} x(t)$, $\sigma \in \{1, \dots, M\}$
is universally stable if & only if $\exists$ "common" Lyapunov function.

Thm: Let $G(t) = (V, E(t))$ (undirected)

$$\text{Then } \dot{x}_i = \sum_{j \in N_i(t)} (x_j - x_i) \quad ; \quad i = 1, \dots, N$$

drives all x_i to $\frac{1}{N} \sum_{j=1}^N x_j(0)$ if $G(t)$ is connected at all times.

Proof: Take $V(x) = \frac{1}{2} x^T x$ as Lyapunov function

$$\dot{V}(x) = -x^T L_{\sigma(t)} x$$

$\dot{V}$ vanishes on $\text{span}\{1\}$ for all modes.

Time varying Graphs:

$$E_t^\tau = \bigcup_{s=t}^{\tau} E(s) \quad [\text{Union Edgeset}]$$

$$G_t^\tau = (V, E_t^\tau) \quad [\text{Union Graph}]$$

Def: $G(t)$ is jointly connected if $\exists \Delta T < \infty$
 s.t. $G_t^{t+\Delta T}$ is connected $\forall t$.

Theorem: The algorithm $\dot{x} = -L_{\sigma(t)} x$ will achieve average consensus
 $x(t) \rightarrow \frac{1}{N} (x(0)^T \mathbf{1}) \mathbf{1}$ iff $G(t)$ is jointly connected.

Ex: Δ -disk proximity graph.

$\{i, j\}$ is an edge iff $\|x_i - x_j\| \leq \Delta$

Assume $W > 0$
 κ degree matrix

$$L_W = D W D^T = (D W^{1/2}) (D W^{1/2})^T$$

$$V = \frac{1}{2} x^T x$$

$$\dot{V} = -x^T L_W x \leq 0$$

$$\Rightarrow \text{if } \dot{V} = 0 \quad (\Rightarrow) W^{1/2} D^T x = 0$$

$$(\Rightarrow) x \in \text{span}(\mathbf{1})$$

What if weights depend on x ? $\{W(x)\}$

$$\dot{x} = -D W(x) D^T x$$

$$V(x) = \frac{1}{2} x^T x$$

$$\dot{V}(x) = -x^T (D W(x) D^T) x \leq 0$$

$$= 0 \text{ iff } x \in \text{span}(\mathbf{1})$$

Total Network Energy

$$\hookrightarrow E(x) = \frac{1}{2} \sum_{i=1}^N \sum_{j \in N_i} E_{ij}(\|x_i - x_j\|)$$

$$E_{ij}(\|x_i - x_j\|) \geq 0$$

$$= 0 \text{ iff } \|x_i - x_j\| = 0$$

$$\text{Let } \dot{x} = - \frac{\partial E(x)}{\partial x}$$

$$\text{Then } \dot{E}(x) = - \left\| \frac{\partial E}{\partial x} \right\|^2 \leq 0$$

$$\begin{aligned} \dot{x}_i &= - \frac{\partial E}{\partial x_i} = - \frac{\partial}{\partial x_i} \left(\frac{1}{2} \sum_{i=1}^N \sum_{j \in N_i} E_{ij}(\|x_i - x_j\|) \right) \\ &= - \frac{\partial}{\partial x_i} \left(\sum_{j \in N_i} E_{ij}(\|x_i - x_j\|) \right) \end{aligned}$$

$$= \sum_{j \in N_i} \left. \frac{\partial \varepsilon_{ij}(y)}{\partial y} \right|_{y = \|x_i - x_j\|} \cdot \frac{(x_i - x_j)}{\|x_i - x_j\|}$$

$y = \|x_i - x_j\|$

$$\|v\| = (v^T v)^{1/2}$$

$$\frac{\partial \|v\|}{\partial v} = \frac{1}{2} (v^T v)^{-1/2} (2v) = \frac{v}{\|v\|} = - \sum_{j \in N_i} w_{ij}(x) \cdot (x_i - x_j)$$

Ex 1 $\varepsilon_{ij} = \frac{1}{2} \|x_i - x_j\|^2 \rightarrow \mathcal{E} = \frac{1}{2} x^T L x$

$$\frac{\partial \varepsilon_{ij}}{\partial x_i} = (x_i - x_j)$$

$$\dot{x}_i = - \sum_{j \in N_i} \varepsilon (x_i - x_j) \leftarrow \text{Regular consensus}$$

Midterm syllabus until formation control

If graph depends on proximity: Δ -Disk graph
 $\hookrightarrow$ connectivity can break.

Ex 2 $\varepsilon_{ij}(\|x_i - x_j\|) = \|x_i - x_j\|$

$$\frac{\partial \varepsilon_{ij}}{\partial x_i} = \frac{(x_i - x_j)}{\|x_i - x_j\|} \Rightarrow w_{ij} = \frac{1}{\|x_i - x_j\|}$$



Make energy go to ∞ around Δ .

Make controller s.t. $\dot{\varepsilon}(x) \leq 0$

$$\varepsilon_{ij} = \frac{\|x_i - x_j\|^2}{\Delta - \|x_i - x_j\|}$$

$$\frac{\partial \varepsilon_{ij}}{\partial y} = \frac{2y}{\Delta - y} + \frac{y^2}{(\Delta - y)^2}$$

$$= \frac{2y\Delta - 2y^2 + y^2}{(\Delta - y)^2} \rightarrow \frac{y(2\Delta - y)}{(\Delta - y)^2} \cdot \frac{(x_i - x_j)}{y}$$

$$w_{ij} = \frac{2\Delta - y}{(\Delta - y)^2}$$

Issue: agent entering Δ , then will lead to ∞ velocity.

Introduce switch: $h(y) = \begin{cases} 1 & \text{if } y < \varepsilon \\ 0 & \text{else} \end{cases}$

$$\dot{x}_i = - \sum_{j \in N_i(t)} \frac{2\Delta - \|x_i - x_j\|}{(\Delta - \|x_i - x_j\|)^2} (x_i - x_j) h(\|x_i - x_j\|)$$

Theorem: Let $G(t)$ be a Δ disk graph initially connected with

$$\|x_i(0) - x_j(0)\| \leq \Delta - \varepsilon \quad \forall (v_i, v_j) \in E(0)$$

then (↑) consensus algorithm:

- (i) Edges are never lost & only add, $G(t)$ remains connected
- (ii) $x_i \rightarrow \frac{1}{N} \sum_{j=1}^N x_j(0)$

Double Integrator Dynamics

$$\ddot{p}_i = u_i$$

Assume SISO systems: $x_i = \begin{bmatrix} p_i \\ \dot{p}_i \end{bmatrix} \quad \dot{x}_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i$

$$y_i = \begin{bmatrix} 1 & 0 \end{bmatrix} x_i$$

Goal: Choose rendezvous-type algorithm on inputs that achieves consensus on outputs:

→ N copies of this system

→ Identical copies, same $\hat{A}, \hat{B}, \hat{C}$

Approach: Given undirected graph G ,

$$\begin{aligned} \text{set } u_i &= \sum_{j \in N_i} (y_j - y_i) \\ &= \hat{C} \sum_{j \in N_i} (x_j - x_i) \end{aligned}$$

$$\dot{x}_i = \hat{A} x_i + \hat{B} \hat{C} \sum_{j \in N_i} (x_j - x_i)$$

Example Single integrator proportional gain

$$\dot{x}_i = -k_p x_i + u_i, \quad y_i = x_i$$

$$u_i = \sum_{j \in N_i} (x_j - x_i)$$

$$\hat{A} = -k_p$$

$$\hat{B} = 1$$

$$\hat{C} = 1$$

$$\dot{x} = -k_p x - Lx$$

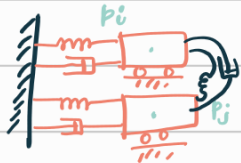
↳ Eigenvalues: λ_i of $L \rightarrow$ new: $-\lambda_i - k_p$

↳ consensus to 0.

Asymptotically stable.

2nd Order system:

$$\ddot{p}_i = \underbrace{-k_p p_i}_{\text{spring}} - \underbrace{k_d \dot{p}_i}_{\text{damp}} + u_i$$



$$u_i = \sum_{j \in N_i} r_p (p_j - p_i) + r_d (\dot{p}_j - \dot{p}_i)$$

Coupled Linear System formulation

$$x_i = \begin{bmatrix} p_i \\ \dot{p}_i \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\hat{C} = [r_p \quad r_d]$$

$$(y_i = r_p p_i + r_d \dot{p}_i)$$

$$u_i = \sum_{j \in N_i} (y_j - y_i)$$

Ensemble

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

$$\dot{x} = (I_N \otimes \hat{A}) x + (I_N \otimes \hat{B}) u$$

$$\begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$$

$$y = (I_N \otimes \hat{C}) x$$

$$u = -Ly$$

Diffusively Linear System

N-SISO linear systems

$$\dot{x}_i = \hat{A} x_i + \hat{B} u_i$$

$$x_i \in \mathbb{R}^d, u_i \in \mathbb{R}, y_i \in \mathbb{R}$$

$$y_i = \hat{C} x_i$$

$$u_i = - \sum_{j \in N_i} (y_i - y_j)$$

$$x \in \mathbb{R}^{dN}$$

$$y \in \mathbb{R}^N$$

$$u \in \mathbb{R}^N$$

$$\dot{x} = (I_N \otimes \hat{A}) x + (I_N \otimes \hat{B}) u$$

$$y = (I_N \otimes \hat{C}) x$$

$N \times N_d$

$$u = -Ly$$

↑
Laplacian

$$\begin{aligned}\dot{x} &= (\mathbf{I}_N \otimes \hat{A})x + (\mathbf{I}_N \otimes \hat{B})(-L)(\mathbf{I}_N \otimes \hat{C})x \\ &= (\mathbf{I}_N \otimes \hat{A} - L \otimes \hat{B}\hat{C})x\end{aligned}$$

Mixed Product Property of $\otimes$: $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

Synchronization Thm:

Let $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N$ be

eigenvalues of L from undirected connected G .

Then the states synchronize in the sense:

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_j(t)\|_2 \rightarrow 0 \quad \forall i, j$$

if & only if $\hat{A} - \lambda_i \hat{B}\hat{C}$ is Hurwitz
 $\forall i = 2, \dots, N$

In this case, states converge to solution of unforced system w/any initial condition:

$$\dot{\bar{x}} = \hat{A} \bar{x}, \quad \bar{x}(0) = \frac{1}{N} \sum_{i=1}^N x_i(0)$$

If we take $\hat{A}=0, \hat{B}=1, \hat{C}=1$

$\hookrightarrow$ we get the avg. consensus.
 $-\lambda_i < 0 \quad \forall i = 2, \dots, N$

Synchronization Specialized to 2nd order system:

$$\hat{A} - \lambda_i \hat{B}\hat{C} = \begin{bmatrix} 0 & 1 \\ -k_p - \lambda_i r_p & -k_d - \lambda_i r_d \end{bmatrix}$$

Aside: $\begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$ is Hurwitz if $a < 0, b < 0$

(Under $k_p, k_d, r_p, r_d \geq 0$)

$$\Rightarrow k_p + \lambda_i r_p > 0 \quad \& \quad k_d + \lambda_i r_d > 0$$



(if) $(k_p, r_p \text{ or both } > 0)$ and
 $(k_d, r_d \text{ or both } > 0)$

($\Rightarrow$) $(k_p + r_p) > 0$ and $(k_d + r_d) > 0$.

Thm

Consider

$$\ddot{p}_i = -k_p r_i - k_d \dot{p}_i + u_i$$

$$u_i = \sum_{j \in N_i} (r_p (p_j - p_i) + r_d (\dot{p}_j - \dot{p}_i))$$

$$k_p, k_d, r_p, r_d \geq 0$$

$$\text{if } k_p + r_p > 0 \text{ \& } k_d + r_d > 0$$

then synchro. is achieved.

$$\lim_{t \rightarrow \infty} \begin{cases} \|p_i(t) - p_j(t)\| = 0 \\ \|\dot{p}_i - \dot{p}_j(t)\| = 0 \end{cases}$$

Converge to what?

$$\text{Avg. system: } \begin{bmatrix} \dot{\bar{p}} \\ \dot{\bar{v}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix} \begin{bmatrix} \bar{p} \\ \bar{v} \end{bmatrix}$$

$$\begin{bmatrix} \bar{p}(0) \\ \bar{v}(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{N} \sum p_i(0) \\ \frac{1}{N} \sum v_i(0) \end{bmatrix}$$

What if $k_p < 0$ and/or $k_d < 0$ but still $k_p + \lambda_i r_p > 0 \text{ \& } k_d + \lambda_i r_d > 0$
* $i = 2, \dots, N$?

Case 1:

Pure integrator case:

$$k_p = 0, k_d = 0$$

Synchronization condition $r_p > 0 \text{ \& } r_d > 0$

$$\text{Average system: } \ddot{\bar{p}} = 0$$

$$\bar{p}(t) = \bar{p}(0) + t \bar{v}(0)$$

Case 2:

$$k_p = 0, r_p > 0$$

$$r_d = 0, k_d > 0$$

Springs between agents
* damping from ambient "fluid".



$$\dot{\bar{p}} = \bar{v}$$

$$\dot{\bar{v}} = -k_d \bar{v}$$

$$v(t) = e^{-k_d t} \bar{v}(0)$$

$$\bar{p}(t) = \left(\frac{1}{k_d} - \frac{e^{-k_d t}}{k_d} \right) \bar{v}(0) + \bar{p}(0)$$

$$\ddot{p}_i = u_i$$

$$u_i = \sum_{j \in N_i} (r_j (p_j - p_i) + r (\dot{p}_j - \dot{p}_i))$$

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \hat{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \hat{C} = [1 \quad r]$$

$$x = \begin{bmatrix} p_1 \\ v_1 \\ p_2 \\ v_2 \\ \vdots \\ p_N \\ v_N \end{bmatrix}$$

$$\dot{x} = I_N \otimes \hat{A} x + I_N \otimes \hat{B} u$$

$$y = I_N \otimes \hat{C} x$$

$$u = -Ly$$

$$\dot{x} = (I_N \otimes \hat{A} - L \otimes \hat{B} \hat{C}) x$$

Theorem: Eigenvalues of $I_N \otimes \hat{A} - L \otimes \hat{B} \hat{C}$ are Eigenvalues of $\hat{A} - \lambda_i \hat{B} \hat{C}$ for λ_i an eigenvalue of L .
(undirected & connected L)

Proof: L is symmetric, write $L = U \Lambda U^T$
 $\tilde{\Lambda}$ Eigenvalues of L , U is orthonormal matrix

$$z = (U^T \otimes I_2) x$$

$$\{(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)\}$$

$$(U^T \otimes I_2) \cdot (U \otimes I_2) = I_{2N}$$

$$\Rightarrow (U^T \otimes I_2)^{-1} = (U \otimes I_2)$$

$$\begin{aligned} \dot{z} &= (U^T \otimes I_2) (I_N \otimes \hat{A} - L \otimes \hat{B} \hat{C}) (U \otimes I_2) z \\ &= ((U^T I_N U \otimes I_2 \hat{A} I_2) - U^T L U \otimes (\hat{B} \hat{C})) z \\ &= (I_N \otimes \hat{A} - \Lambda \otimes \hat{B} \hat{C}) z \end{aligned}$$

$$\dot{z} = \begin{bmatrix} \hat{A} - \lambda_1 \hat{B} \hat{C} & 0 & 0 & 0 \\ 0 & \hat{A} - \lambda_2 \hat{B} \hat{C} & 0 & \vdots \\ 0 & 0 & \hat{A} - \lambda_3 \hat{B} \hat{C} & \vdots \\ 0 & \dots & 0 & \hat{A} - \lambda_n \hat{B} \hat{C} \end{bmatrix} z$$

■

$$\begin{aligned} \text{Eigenvalues of } \hat{A} - \lambda_i \hat{B} \hat{C} &\rightarrow x^2 + r\lambda_i x + \lambda_i \\ \begin{bmatrix} 0 & 1 \\ -\lambda_i & -r\lambda_i \end{bmatrix} &= \frac{-r\lambda_i \pm \sqrt{r^2\lambda_i^2 - 4\lambda_i}}{2} \end{aligned}$$

For stability, $r > 0$

$$r^2 \leq \frac{4}{\lambda_i} \Rightarrow \text{complex conjugate eigenvalues in LHP.}$$

$$r^2 > \frac{4}{\lambda_i} \Rightarrow 2 \text{ real eigenvalues}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \leftarrow \text{is unstable}$$

Thm: The agreement protocol:

$\dot{x} = -Lx$ over digraph with directed Laplacian L achieves
 average consensus iff balanced & weakly connected
 if balanced; weakly connected ($\Rightarrow$) strongly connected.

Pf: Assume balanced & weakly:

Construct a "circuit", directed path starts where it ends
 (path, visit multiple times but edge once)

does not pass an edge twice. Start somewhere & start walking.
 By balanced, this continues until back where I started.
 Now remove, resulting graph is still balanced.

$$\dot{x} = -Lx$$

$$q \text{ st } q^T L = 0$$

$$q^T \dot{x} = 0 \Rightarrow \lambda(A) = \frac{q^T x(0)}{q^T \mathbf{1}} ; \quad q = \mathbf{1} \text{ iff } L \text{ is balanced.}$$

Formation Control

Consensus in terms of Equilibrium.

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^{Nd}$$

$$\mathcal{X} = \left\{ \begin{bmatrix} \tau \\ \tau \end{bmatrix} \in \mathbb{R}^{Nd} : \tau \in \mathbb{R}^d \right\} = \left\{ \mathbf{1}_N \otimes \tau : \tau \in \mathbb{R}^d \right\} \subset \mathbb{R}^{Nd}$$

$\hookrightarrow$ Set of acceptable Equilibria

Consensus means: $x(t) \rightarrow \mathcal{X}$, as $t \rightarrow \infty$.

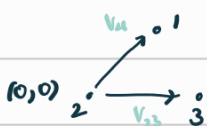
Define a **FORMATION**:

$\mathcal{X} \subset \mathbb{R}^{Nd}$ is a formation,

$$\text{if } x \in \mathcal{X} \Rightarrow (x + \mathbf{1}_N \otimes \tau) \in \mathcal{X} \quad \forall (\tau \in \mathbb{R}^d)$$

($\hookrightarrow \mathcal{X}$ is non empty)

Example: $\mathcal{X} = \left\{ \xi + \mathbf{1}_N \otimes \tau \mid \tau \in \mathbb{R}^d \right\}$



$$\xi = \begin{bmatrix} v_{21} \\ 0 \\ v_{23} \end{bmatrix}$$

Scale invariance: $x \in \mathcal{X} \Rightarrow \alpha x \in \mathcal{X} \quad \forall \alpha > 0$.

Example: $\mathcal{X} = \left\{ \begin{bmatrix} \alpha v_{21} + z \\ z \\ \alpha v_{23} + z \end{bmatrix} : \alpha > 0, z \in \mathbb{R}^d \right\}$

Formations are usually specified using relative displacement / distances.

Let $G = (V, E)$ be directed graph.

Let $\delta_{(i,j)} \in \mathbb{R}^d$ be desired displacement $(i,j) \in E$

$$\mathcal{X}(\Delta) \rightarrow \mathcal{X} = \{x \in \mathbb{R}^{nd} : x_j - x_i = \delta_{(i,j)}\}$$

if non empty, then formation is FEASIBLE.

$$\mathcal{X}(\alpha \Delta) \rightarrow \mathcal{X} = \{x \in \mathbb{R}^{nd} : \exists \alpha \text{ s.t. } x_j - x_i = \alpha \delta_{(i,j)} \quad \forall (i,j) \in E\}$$

}
 using displacements.

Using distances: $G = (V, E)$, undirected graph

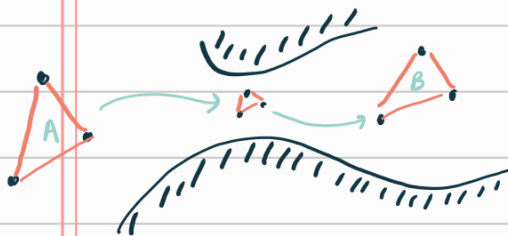
$$D = \{d_{ij}\}_{i,j \in E}, \quad d_{ij} > 0$$

$$\mathcal{X} = \{x \in \mathbb{R}^{nd} : \|x_i - x_j\|_2 = d_{ij} \quad \forall d_{ij} \in D\}$$

D is feasible if $\mathcal{X}(D)$ is non empty.

x is relationally invariant if x has form

$$\mathcal{X} = \{x \in \mathbb{R}^{nd} : \|x_i - x_j\|_2 = \alpha d_{ij} \quad \forall d_{ij} \in D, \text{ for some } \alpha > 0\}$$



Multirobot team achieves a formation x if $x(t) \rightarrow x, t \rightarrow \infty$

A first candidate for formation control

Let x be a formation

Let $\bar{x} \in \mathcal{X}$

$$\bar{\delta}_{(i,j)} = \bar{x}_j - \bar{x}_i \quad \text{for directed } G = (V, E)$$

$$\dot{x}_i = - \sum_{j \in N_i^{\text{in}}} (x_i - x_j - \bar{\delta}_{(j,i)})$$

$$= - \sum_{j \in N_i^{\text{in}}} (x_i - x_j - (\bar{x}_i - \bar{x}_j)) = - \sum_j ((x_i - \bar{x}_i) - (x_j - \bar{x}_j))$$

$$\text{Let } \hat{x}_i = x_i - \bar{x}_i \Rightarrow \dot{\hat{x}} = -L \hat{x}$$

Directed Consensus if L has R.O.T.

$$\Rightarrow \hat{x} = \mathbb{1} \otimes z \Rightarrow x = \bar{x} + \mathbb{1} \otimes z \in \mathcal{X} \quad \exists z \in \mathbb{R}^d$$

Big problem: $\rightarrow$ Have to commit to particular $\bar{x}$

$\rightarrow$ No way to exploit rotational freedom.

$\rightarrow$ Our goal, then, is a different algorithm that is on distances.

$\rightarrow$ Rigidity of formation

Assignment-free Solution

$$\mathcal{S}: \{1, \dots, N\} \rightarrow \{1, \dots, N\}$$

$\uparrow$ Permutation γ bijection.

$$X \text{ is assignment free: } \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in X \Rightarrow \mathcal{S} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in X \quad \forall \mathcal{S} \in \text{Permutation}$$

Distance-based formation control:

$\hookrightarrow$ Need to understand rigidity

$\hookrightarrow$ will only have nonlinear control scheme for local convergence.

Let suppose we have a formation given by:

$$X = \{x \in \mathbb{R}^{Nd}: \|x_i - x_j\| = d_{ij}, \forall \{i, j\} \in E\}$$

For some $G = (V, E)$, where $\{d_{ij}\}_{\{i, j\} \in E}$ is a set of desired distances.

We want formation control:

$\hookrightarrow$ Decentralized

$\hookrightarrow$ Few edges

$\hookrightarrow$ Not "long" edges.

Formation Graph: $G_f = (V, E, W)$

$w(e) \in W \leftarrow$ desired length of edge e .

(?)

Set of feasible points $\Xi = \{\xi_1, \dots, \xi_n\}; \xi_i \in \mathbb{R}^d$

$$\text{s.t. } \|\xi_i - \xi_j\| = w(e) \quad \forall e(i, j) \in E.$$

Framework: is a pair $(\Xi, G_f) = G(\Xi)$

Trajectory of a framework is a set of continuous functions $x_1(t), \dots, x_n(t)$ s.t. $x(0) = \Xi$

A trajectory is **edge-consistent** if it maintains correct edge distances for all edges in graph,

$$\|x_i(t) - x_j(t)\| = w(e) = \|x_i - x_j\| \quad \forall \{i, j\} \in E$$

Framework is **rigid**: if all **edge-consistent** trajectories also **maintain pairwise distances**.

$$\|x_i(t) - x_j(t)\| = \|x_i(\tau) - x_j(\tau)\| \quad \forall t, \tau, i, j$$

→ Rigidity depends on dimension "d"

→ Does not preclude multiple equilibria

How to check if Ξ is rigid?

Rigidity depends only on formation graph, not specific feasible point.

→ Can talk about whether formation graphs G_f are rigid.

Edge consistency $\Rightarrow (x_i - x_j)^T (x_i - x_j) = \text{const.}$
 derivative $\Rightarrow (\dot{x}_i - \dot{x}_j)^T (x_i - x_j) = 0$

If we can show that this further implies $(\dot{x}_i - \dot{x}_j)^T (x_i - x_j) = 0 \quad \forall i, j$ then we have rigidity.

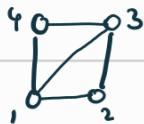
Def: A set of velocities

$$\dot{x}_i = u_i, \text{ s.t.}$$

$$(u_i - u_j)^T (\dot{x}_i - \dot{x}_j) = 0 \quad \forall \{i, j\} \in E$$

is an **infinitesimal motion** of framework Ξ .

Ex:



Edge-consistent trajectory

$$\begin{bmatrix} (x_1 - x_2)^T & (x_1 - x_3)^T & 0 & 0 \\ (x_1 - x_3)^T & 0 & -(x_2 - x_3)^T & 0 \\ 0 & \sim & \sim & 0 \\ 0 & 0 & \sim & \sim \\ \sim & 0 & 0 & \sim \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = 0$$

Rigidity matrix = $R(x, G_f)$

* Write the potential function & compute the null direction

Edge-continuity requires that $R(x) \cdot u = 0$

↳ For rigidity, we want only remaining local infinitesimal motions correspond to translations & rotations.

For example: in $\mathbb{R}^2$ we want 3 degrees of freedom, we want null space of $R(x)$ to only have 3-dimension (2D).

• Nullspace of Rigidity matrix corresponds to allowed velocities.

Rank $(R(x)) = 2N - 3$ • If all allowed velocities are trivial, we are rigid.
 $\{2N = ? + 3\}$

Framework is infinitesimally rigid if:

null space of $R(x) = \#$ of degrees of rot & translation $\mathbb{R}^d$

for example $d=2 \Rightarrow 3$
 $d=3 \Rightarrow 6$

Equivalent Rank condition:

$$\text{rank}(R(x, G_f)) = \begin{cases} 2N-3 & \text{if } d=2 \\ 3N-6 & \text{if } d=3 \end{cases}$$

Then: Infinitesimal Rigidity $\Rightarrow$ Rigidity

Rigidity $\nRightarrow$ Infinitesimal Rigid

Rigid but not infinitesimally rigid:



A graph G is generically rigid if exists a framework for the graph that is infi rigid.
 has position information

Example:

$$R(x, G_f) = \begin{bmatrix} (x_1 - x_2)^T & -(x_1 - x_2)^T & 0 \\ 0 & (x_2 - x_3)^T & -(x_1 - x_3)^T \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\text{Rank}(R) = 2$$

We need Rank $= 2 \times 3 - 3 = 3$ for Rigid formation.

Minimally Rigid: A graph G_f is minimally rigid if it is rigid & removing any edge renders the graph not rigid.

checking minimal rigidity is hard.

Distance-based formation control

Leader-follower network

→ set of desired interagent distances

$$D = \{d_{ij} : \{i, j\} \in E_F\}$$

→ Need formation to be rigid

Rigidity is hard to check, so we check infinitesimal rigidity, using the rigidity matrix. $\rightarrow Ru = 0$ only has soln u (infinitesimal motion)

Distance-based control scheme: $E(x) = \frac{1}{2} \sum_{i=1}^N \sum_{j \in N_i} E_{ij}(x_i, x_j)$

$$\dot{x}_i = -\frac{\partial E}{\partial x_i}(x) = -\sum_{j \in N_i} \frac{\partial E_{ij}}{\partial x_i}(x_i, x_j)$$

$$E_{ij}(x_i, x_j) = (\|x_i - x_j\| - d_{ij})^2$$

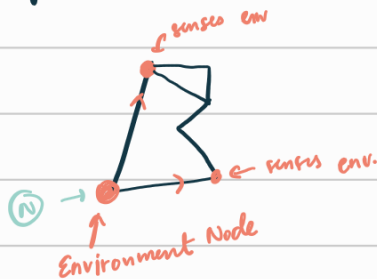
↑ Energy which is minimized when $\|x_i - x_j\| = d_{ij}$

$$\dot{x}_i = -\sum_{j \in N_i} \underbrace{2(\|x_i - x_j\| - d_{ij})}_{\text{weight}} \cdot \frac{(x_i - x_j)}{\|x_i - x_j\|}$$

$$E_{ij}(x_i, x_j) = (\|x_i - x_j\|^2 - d_{ij}^2)^2$$

$$\dot{x}_i = -\sum_{j \in N_i} 4(\|x_i - x_j\|^2 - d_{ij}^2)(x_i - x_j)$$

Some agents can measure the environment:



$$\dot{x}_i = \sum_{j \in N_i} (x_j - x_i), \quad i \neq N$$

$$\dot{x}_N = 0$$

(scales)

$$x_f = \begin{bmatrix} x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \dot{x}_f \\ \dot{x}_N \end{bmatrix} = -\begin{bmatrix} L_f & l \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_f \\ x_N \end{bmatrix}$$

$$x_N$$

$$\Rightarrow \dot{x}_f = -\underbrace{L_f}_{(N-1) \times (N-1)} x_f - \underbrace{l}_{(N-1) \times 1} x_N$$

Thm: If G is connected (assume undirected, unweighted), then L_f (derived from the corresponding graph Laplacian) is positive definite, i.e. $L_f > 0$

Pf: by contradiction: Assume $\exists \eta \neq 0$, s.t. $\eta^T L_f \eta = 0$
then show that null space of L is larger than span (1)

$$\dot{x} = Ax + b$$

Suppose A is Hurwitz (all eigenvalues in left half plane)

Then: A is invertible $\rightarrow x^* = -A^{-1}b$ is an equilibrium
 $\uparrow$ Globally Exponentially Stable.

leader follower problem:

$$L_f > 0, \quad -L_f \leftarrow \text{Hurwitz}$$

$$\hookrightarrow x_f^e = -L_f^{-1} l x_N$$

$\hookrightarrow$ GES eq for ANCHORED consensus problem.

Finding the Equilibrium: $\dot{x}_i = 0$

$$\Rightarrow \deg(i) \cdot x_i^e = \sum_{j \in N_i} x_j^e$$

$$\Rightarrow x_i^e = \frac{1}{|N_i|} \sum_{j \in N_i} x_j^e \quad \forall i \neq N.$$

$\hookrightarrow$ In convex hull of n'hood. A centroid of its neighbourhood.

Only possible if everybody at same point.

Then: Quorum $\dot{x}_f = -L_f x_f - l x_N \leftarrow$ connected, undirected graph
 (one anchor node case) and $\dot{x}_N = 0$, then $x_f \rightarrow \frac{1}{N-1} x_N$.

Multiple leaders:

$$x_f = \begin{bmatrix} x_1 \\ \vdots \\ x_{N-N_f} \end{bmatrix} \quad x_L = \begin{bmatrix} x_{N-N_f+1} \\ \vdots \\ x_N \end{bmatrix}$$

$$L = \begin{bmatrix} L_f & \vdots & l \\ \vdots & \ddots & \vdots \\ l^T & \vdots & \lambda \end{bmatrix}$$

$$L_f : (N-N_L) \times (N-N_L)$$

$$l : (N-N_L) \times N_L$$

$$l^T : N_L \times (N-N_L)$$

$$\lambda : N_L \times N_L$$

$$\dot{x}_f = -L_f x_f - l x_L$$

Ex: $\begin{matrix} 3 & 0 & 1 & 2 & 0 & 4 \\ \circ & - & \circ & - & \circ & - & \circ \\ \dot{x}_3 = 0 & & \dot{x}_4 = 1 & & \dot{x}_5 = 0 & & \dot{x}_6 = 0 \end{matrix}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

$-L_f \quad \quad -l$

$$\hookrightarrow x_f^e = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

Suppose a network of mobile robots being controlled by leader agents.

$$\dot{x}_i = \sum_{j \in N_i} (x_j - x_i) \quad ; \quad i \in \{1, \dots, N_f\}$$

$$\dot{x}_i = x_i \quad ; \quad i \in \{N_f+1, \dots, N\}$$

$$L = \begin{bmatrix} L_f & l \\ l^T & \lambda \end{bmatrix}$$

$$\dot{x}_f = -L_f x_f - l x_L$$

$$\dot{x}_L = x_L$$

From velocity control, we can assume to directly control x_L .

$$\hookrightarrow \dot{x}_f = \underbrace{-L_f}_{A} x_f - \underbrace{L}_{B} \underbrace{v}_{u}$$

Controllability of LTI systems: $\exists$ input signal $u(t)$ to drive $\dot{x} = Ax + Bu$ from arbit. $x(0)$ to $x(t)$ in any time t .

Controllability matrix: $\Gamma = [B \quad AB \quad \dots \quad A^{n-1}B]$

$\uparrow$ Full rank for controllability.

Leader follower Networks

N agents

$1, \dots, N_f \leftarrow$ followers

$N_{f+1}, \dots, N \leftarrow$ leaders

$$x_f = \begin{bmatrix} x_1 \\ \vdots \\ x_{N_f} \end{bmatrix}, x_L = \begin{bmatrix} x_{N_f+1} \\ \vdots \\ x_N \end{bmatrix}$$

Problem 1: $\dot{x}_i = 0$

$i = N_{f+1}, \dots, N$

$$\dot{x}_i = \sum_{j \in N_i} (x_j - x_i) \quad i = 1, \dots, N_f$$

$$L = \begin{bmatrix} L_f & \vdots & l \\ \vdots & e^T & \vdots \\ 0 & \vdots & \lambda \end{bmatrix}$$

$$L_f: N_f \times N_f$$

$$l: N_f \times (N - N_f)$$

$$\lambda: (N - N_f) \times (N - N_f)$$

Theorem: If G is connected, undirected L_f is PD, $L_f > 0$

$\Rightarrow -L_f$ is Hurwitz

$$\Rightarrow x_f^c = -L_f^{-1} l x_L(0)$$

Convex hull
of leader positions

Let's treat leaders as control inputs.

Maybe we control x_L , $\leftarrow$ basically we control $x_L(t)$

$$\text{Set } u(t) = x_L(t)$$

$$\dot{x}_f = -L_f x_f - l u$$

How much control authority do I have here?

$\hookrightarrow$ use controllability.

REFRESHER: $\dot{x} = Ax + Bu$

$\exists? u(t)$ s.t. x_0 to x_f in exactly T time.

$$\Gamma = \begin{bmatrix} A & AB & \dots & A^{n-1}B \end{bmatrix} \begin{matrix} n \times n_i \end{matrix}$$

Controllable subspace is range space of Γ :

$$R(\Gamma) = \{ x: \exists y \quad x = \Gamma y \}$$

$$\dot{x} = Ax + Bu$$

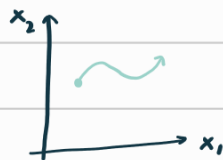
Lemma: Possible to control from $x(0)$ to x_T in time T
 if & only if $(x_T - e^{AT} x_0) \in R(\Gamma)$

Lemma: if $x_0, x_T \in R(\Gamma)$, then you can steer b/w these 2 points
 in arbitrary amt. of time.

Lemma: if $R(\Gamma) = \mathbb{R}^n$, then we can control b/w ^{any} x_0 & x_f in any amt.
 of time. we call $(A, B) \leftarrow$ completely controllable

Double Integrator $\ddot{x}_1 = u$ $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
 $\Gamma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Well, can I exactly follow arbitrary trajectory $x(t) \forall t$.



Certainly not true for double integrator,
 given desired trajectory $x_1(t), x_2(t)$, could have hope
 for exact tracking if $\dot{x}_1(t) = x_2(t)$

Back to Leader follower:

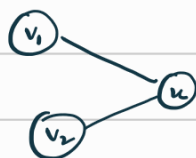
How to structure graph for complete controllability:

$$\dot{x}_f = -L_f x_f - l u$$

$$A = -L_f, B = -l$$

Matrices are constructed from graph $G=(V, E)$

Ex)



$$L = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$L_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$l = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\dot{x}_f = -L_f x_f - l u$$

$$\begin{cases} \dot{x}_1 = -x_1 + u \\ \dot{x}_2 = -x_2 + u \end{cases}$$

Let's check controllability

$$\Gamma = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\leftarrow \text{rank} = 1 < 2$$

$\Rightarrow$ Not completely cont.

Ex:



$$L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} -L & L_f L \\ -L_f^2 L \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \Rightarrow \text{Not comp. cont.}$$

Ex:



$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} -L & L_f L & -L_f^2 L \\ 1 & -1 & 0 \\ 1 & -2 & 6 \\ 0 & 1 & -3 \end{bmatrix} \rightarrow \text{full rank} \Rightarrow \text{complete cont.}$$

To formalize this intuition about symmetries, we introduce equitable partitions:

A partition of $G = (V, E)$ is a grouping of V into cells

Given graph $G = (V, E)$ and partition $\{C_1, \dots, C_p\}$, $C_i \subset V$

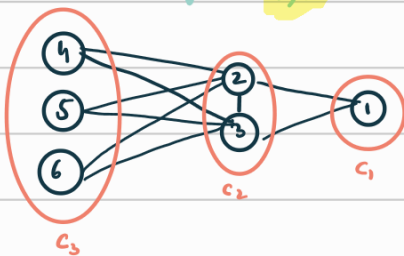
node to cell degree: $\deg(v, C) = \text{card}(\{v' \in C : (v, v') \in E\})$
 (cardinality)

Equitable partition: A partition Π is equitable partition (EP)

if

$$\deg(v, C) = \deg(v', C) \quad \forall C \in \Pi$$

$\forall v, v'$ in same cell.



$$\deg(v_1, C_1) = 0, \quad \deg(v_1, C_2) = 2, \quad \deg(v_1, C_3) = 0$$

(not done EEP)

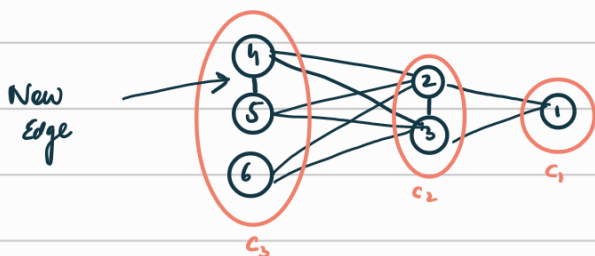
$$\deg(v_2, C_1) = \deg(v_3, C_1) = 1$$

$$\deg(v_2, C_2) = \deg(v_3, C_2) = 1 \leftarrow \text{(not done in EEP)}$$

$$\deg(v_2, C_3) = \deg(v_3, C_3) = 3$$

$$\deg(v_4, C_2) = \deg(v_5, C_2) = \deg(v_6, C_2) = 2$$

rest is 0.



← BREAKS EQUITABLE PARTITION.

Externally Equitable partitions:

Like an EP, except we don't check edges within cells.

$$d_{\pi}(v, C) = d_{\pi}(v', C) \quad \forall C \in \Pi$$

$\forall v, v'$ in same cell & not in C.

- 2 trivial EEPs:
- (i) $C_i = \{v_i\}$
 - (ii) $C = V$

Controllability of LF network

N_f followers

$N - N_f$ leaders

$$\dot{x}_i = \sum_{j \in N_i} (x_j - x_i), \quad x_i \in \mathbb{R}^d$$

$$\dot{x}_i = u_i, \quad i = N_f + 1, \dots, N$$

$$\dot{x}_f = -L_f x_f - l u$$

$$\Gamma = \begin{bmatrix} -l & L_f l & -L_f l & \dots \end{bmatrix}$$

Graph theoretic interpretation $\leftarrow$ symmetry breaks controllability

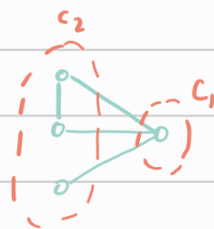
$\hookrightarrow$ partition $\Pi = \{C_1, \dots, C_p\}, \quad C_i \subset V$

$$C_i \cap C_j = \emptyset, \quad \bigcup_i C_i = V$$

Node to cell dyc

$$d_{\pi}(v, C) = \{ \# \text{ of edges connecting } v \text{ to any cell } v' \in C \}$$

$\rightarrow$ Equitable partition & Externally Equitable partition.



$\leftarrow$ Not equitable, but externally equitable.

Def: An EEP is leader invariant if all leaders ^{belong} to their own cells.

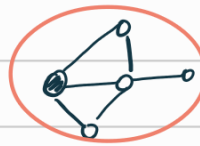
Ex:



Leader Invariant



MAXIMAL



Def Maximal leader invariant EEP is one that has fewest cells of any leader invariant EEP.

There exist a unique maximal leader-invariant EEP.

Necessary CONDITION for COMPLETE CONTROLLABILITY

Thm: A connected LF network is completely controllable only if the maximal leader-invariant EEP is the trivial leader-invariant EEP. $\Rightarrow$

Contrapositive: If maximal $\neq$ trivial $\Rightarrow$ not controllable

Thm: If G is connected & π is the maximal leader-invariant EEP, then,

$$\lim_{t \rightarrow \infty} x_i(t) - x_j(t) = 0$$

For all i, j in same cell.

M is T M is not T

C ✓ ✓ ✗ ✗

NC ✗ ? ✓ ✓

Ex: Line graph N -odd, middle leader

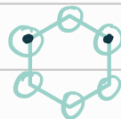


Ex cycle, even N



← Maximal LIEEP

→ Not completely controllable

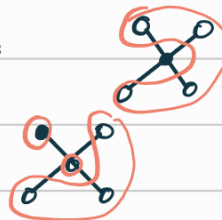


→ completely controllable



← not c.c.

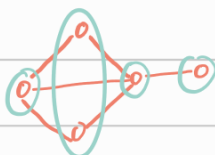
Star:



→ not c.c.

N nodes
↓
need $(n-1)$
leaders for c.c.

Partition π induced a quotient graph (weighted ^vgraph)
Directed



$$\dot{x}_i = \sum (x_j - x_i)$$

$$v_i \in \{1, \dots, N_f\}$$

$$\dot{x}_i = x_i$$

$$i = N_f + 1, \dots, N.$$

Quotient Graph Dynamics: $\dot{z}_i = \sum_{j \in N_{j,i}} \text{deg}(c_j, c_i) (z_j - z_i)$ ← non leader cells c_i

$$\dot{z}_i = u_i \quad i \in \{N_f+1, \dots, N\}$$

$\text{deg}(c_j, c_i) = \# \text{ edges from vertex } j \text{ to } i$

↑ ↑
cells

(One way implication)

Thm If G is connected, Π is maximal leader invariant $\in \mathcal{EP}$, then quotient graph dynamics are completely controllable.

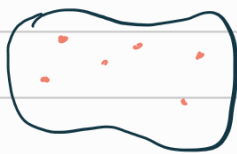
Moreover: $z_i(0) = \frac{1}{|c_i|} \sum_{j \in c_i} x_j(0)$

$$z_i(t) = \frac{1}{|c_i|} \sum_{j \in c_i} x_j(t)$$

- 1) Leader follower system is CC only if the only leader invariant $\in \mathcal{EP}$ is trivial one.
- 2) Controllable subspace is interpreted as the quotient system.
- 3) The uncontrollable part decays asym.
↑ (all states become the same inside cells)

COVERAGE PROBLEM

→ cover an area in some equitable way.



$x_i \in \mathbb{R}^d$ ① → Find locations of robots

② → Partitioning of space

Collection $\{R_1, \dots, R_N\}$ is a tessellation of $\mathbb{R}^d$ if

$$\bigcup_{i=1}^N R_i = \mathbb{R}^d$$

$$\text{int}(R_i) \cap \text{int}(R_j) = \emptyset \quad \forall i \neq j$$

Generally assume $x_i \in R_i$

(a point)

Proximity of Coverage: $q \in R_i$, wrt $x_i \Rightarrow f(\|q - x_i\|) \leftarrow \text{Ex. } \|q - x_i\|^2$

of Region R_i : (a region)

$$S(x_i, R_i) = \int_{R_i} f(\|q - x_i\|) dq = \int_{R_i} \|q - x_i\|^2 dq$$

$$S(x, T) = \sum_{i=1}^N \int_{R_i} |q - x_i|^2 dq$$

Fix x , find optimal tessellation.



Voronoi Regions

$$V_i(x) = \{q \in \mathbb{R}^d : |q - x_i| \leq |q - x_j| \quad \forall j \neq i\}$$

When two Voronoi partitions are adjacent, they are called neighbours. $\leftarrow N(i)$

↳ Can be defined wrt any norm.

↳ For Euclidean norm in $\mathbb{R}^d$, boundaries b/w partitions are $(d-1)$ -dim polytopes (convex)

$T(x) \leftarrow$ Voronoi partition for locations x .

Cost of Tessellation : $S_V(x) = S(x, T(x)) = \sum_{i=1}^N \int_{V_i(x)} |x_i - q|^2 dq$

Goal: Minimize $S_V(x)$ over x .

$$\dot{x}_i = u_i = - \frac{\partial S_V(x)}{\partial x_i}$$

$$\frac{\partial S_V(x)}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\sum_{i=1}^N \int_{V_i(x)} |x_i - q|^2 dq \right)$$

$$= \frac{\partial}{\partial x_i} \left(\sum_{i=1}^N \int_{V_i(x)} |x_i - q|^2 dq \right) \Big|_{x=x} \rightarrow \int_{V_i(x)} \left(\frac{\partial}{\partial x_i} |x_i - q|^2 \right) dq$$

$$+ \frac{\partial}{\partial x_i} \left(\sum_{k=1}^N \int_{V_k(x)} |x_k - q|^2 dq \right) \Big|_{x_i=x_i}$$

$\{$

$\rightarrow = 0$



x_j

* Can we see it from the fact that the Voronoi tessellation is the minimizer so 1st order changes to boundary causes 0 changes?

as equidistant from both agents, the wiggle won't add anything

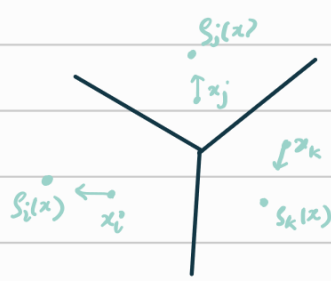
$$- \frac{\partial S_V(x)}{\partial x_i} = - \int_{V_i(x)} \frac{\partial}{\partial x} |x_i - q|^2 dq = -2 \int_{V_i(x)} (x_i - q) dq$$

$$g_i(x) = \frac{\int_{V_i(x)} q dq}{\int_{V_i(x)} dq} = \text{COM of } V_i$$

$$c_i = \frac{1}{2 \int_{V_i(x)} dq} \leftarrow \text{scaling}$$

$$- \frac{\partial S_i}{\partial x_i} \cdot c_i(x) = \boxed{g_i(x) - x_i = u_i}$$

LYONS ALGORITHM for coverage control $u_i = p_i(x) - x_i$



Agents will converge to locations x s.t. $x_i = f_i(s)$.

CENTRAL VORONOI TESSELATION (CVT)

↳ They are local minimizers of converge cost $S_V(x)$.

→ Each agent needs to know its VORONOI neighbours.

Suppose density $\rightarrow \varphi: \mathbb{R} \rightarrow \mathbb{R}$

$$S(x) = \sum_{i=1}^N \int_{V_i(x)} |x_i - q|^2 \varphi(q) dq$$

with same logic:

$$-\frac{\partial S}{\partial x_i}(x) = -2 \int_{V_i(x)} (x_i - q) \varphi(q) dq$$

$$s_i(x) = \frac{\int_{V_i(x)} q \varphi(q) dq}{\int_{V_i(x)} \varphi(q) dq}$$

← weighted COM

$$\hat{x}_i = s_i(x) - x_i$$

converges to a weighted CVT

lets analyse resulting graph of a Voronoi Tesselation.



DELAUNAY GRAPH resulting from locations x .

Delannay triangles:



DOES NOT CONTAIN any other VERTICES

GABRIEL GRAPH is a subgraph of DELAUNAY graph

Edge between i & j iff $\|v_i - v_j\|^2 \leq \|v_i - v_k\|^2 + \|v_j - v_k\|^2$



Discrete - Time Consensus

Ex: Averaging in wireless

↳ Network of agents & independent measurements

→ A distributed algo:

$$x_i^+ = \frac{1}{|N_i|} \sum_{j \in N_i} x_j \quad \text{or} \quad x_i^+ = \frac{1}{|N_i|+1} \left(\sum_{j \in N_i} x_j + x_i \right)$$

Ex: Opinion dynamics

N individuals deciding between options

x_i of prob of persi

$$x_i^+ = \sum_{j=1}^N a_{ij} x_j$$

a_{ij} : Influence
 a_{ii} : confidence in self opinion

Aggregate $x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$

$$x(k+1) = P x(k)$$

$$P_{ij} \geq 0$$

$\alpha \cdot \mathbb{1}$ to be an equilibrium: $P(\alpha \mathbb{1}) = \alpha \mathbb{1}$

$$\Rightarrow P \mathbb{1} = \mathbb{1} \Rightarrow \text{Sum of row of } P = 1$$

Row stochastic Matrices: $\{P \geq 0 \text{ \& } P \mathbb{1} = \mathbb{1}\}$

Column stochastic: $\{P \geq 0 \text{ \& } \mathbb{1}^T P = \mathbb{1}^T\}$

Doubly stochastic: P being row & column stochastic

DOUBLY STOCHASTIC $\Rightarrow$ SYMMETRIC

SYMMETRIC & ROW STOK $\Rightarrow$ DOUBLY STOCHASTIC

Most of results will apply to any row stochastic matrix.

Three ways to construct row stochastic matrices from undirected graph.

METHOD 1

Suppose some undirected $G = (V, E)$, graph Laplacian L .

$$L \mathbb{1} = 0$$

$$P = \mathbb{I} + \alpha L$$

$$\frac{-1}{\max(\deg)}$$

$$-1 < \alpha < 0$$

↳ makes non diagonal true
Enters diagonal true

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

$$-\frac{1}{3} < \alpha < 0$$

Recall C-T consensus: $\dot{x} = -Lx$

Sym

$$x(t+h) = x(t) + h(-Lx(t)) = (I - hL)x(t)$$

$$\{0 < h < \frac{1}{\lambda_2}\}$$

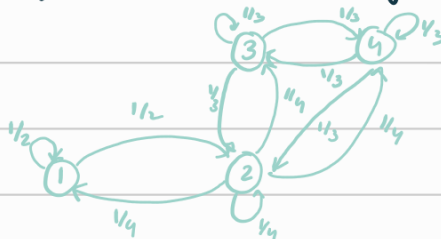
METHOD 2 - Neighbour averaging

$$P_{ij} = \begin{cases} \frac{1}{|N_i| + 1} & \text{if } \{i,j\} \in E \text{ or } i=j \\ 0 & \text{otherwise} \end{cases}$$

Asym



$\Rightarrow$



$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

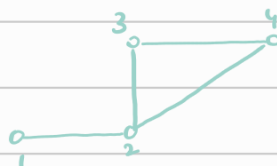
METHOD 3: METROPOLIS HASTINGS

Sym

$$P_{ij} = \begin{cases} \frac{1}{1 + \max(d_i(i), d_i(j))} & \{i,j\} \in E \\ 0 & \{i,j\} \notin E \\ 1 - \sum_{j \neq i} P_{ij} & i=j \end{cases}$$



$\Rightarrow$



$$P = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 5/12 & 1/3 \\ 0 & 1/4 & 1/3 & 5/12 \end{bmatrix}$$

$\leftarrow$ Symmetric & doubly stochastic

Interpreting row stochastic matrices as directed graphs + random walks

$N \times N$ row stochastic P , draw graph with N nodes

edge from j to i with weight P_{ij} .

$\rightarrow$ we can interpret a random walk on this graph.

Def: Row stochastic P is primitive if there exist k s.t. $P^k > 0$
(Every entry of P^k is positive)

$\exists$ exists some # of steps k , starting at any node, positive probability of being at any other node.

Spectrum of row stochastic matrix P :

All other eigenvalues are in the unit disk (Gershgorin Disk Theorem)

$\Rightarrow x(k+1) = P x(k)$ is stable & consensus subspace is equilibria.

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$$

Primitive

$$\frac{x^2 - 1/2 x - 1/2}{2}$$

$$\omega^T P = \omega^T \Rightarrow \omega = \begin{bmatrix} 1 & 1/2 \\ 3/2 \end{bmatrix}$$

$$\text{spec} = \frac{1 \pm 3}{4} = \{-1/2, 1\} \Rightarrow \text{consensus.}$$

Not Primitive

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{spec} = \{1, 1\}$$

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{spec} = \{-1, 1\} \Rightarrow \text{no consensus}$$

Thm: Let P be row stochastic & let $\lambda_0 = 1$ be 1-eigenvalue.

If P is primitive, then $|\lambda_i| < 1$ for all other eigenvalue

$\Rightarrow$ Therefore, for such systems:

$$x(k+1) = P x(k), \quad x(k) \longrightarrow \alpha \mathbf{1}, \text{ for any initial condition } x(0)$$

$$x_i(k) = w^T x(0) \text{ where } w \text{ is the left eigenvector associated to } \lambda_0 = 1$$

$$w^T \mathbf{1} = w^T, \text{ scaled s.t. } \sum_{i=1}^N w_i = 1$$

Let G be any directed graph:

$\hookrightarrow$ A cycle in G is simple if doesn't visit any node more than once.

$\hookrightarrow G$ is periodic if $\exists k > 1$, s.t., length of every simple cycle is a multiple of k .



Periodic with period = 2

$\hookrightarrow G$ is aperiodic if not periodic.

Any self loop makes it aperiodic.

Thm: Let P be a row stochastic matrix. Build directed graph G from P .
 P is **primitive** iff G is **strongly connected** & **aperiodic**.

Example:

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix} \quad P^2 = \begin{bmatrix} 1/4 & 3/4 \\ 0 & 1 \end{bmatrix} \quad P^n = \begin{bmatrix} (1/2)^n & 1 - (1/2)^n \\ 0 & 1 \end{bmatrix}$$

$$\lambda = 1, 1/2$$

It is not PRIMITIVE, but still consensus!!

It is sufficient, not necessary.

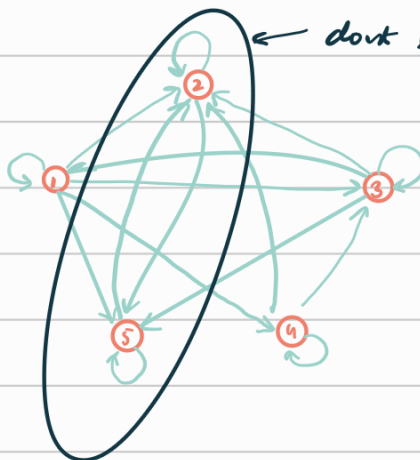
$$P = \begin{bmatrix} 0.15 & 0.15 & 0.1 & 0.2 & 0.4 \\ 0 & 0.55 & 0 & 0 & 0.45 \\ 0.3 & 0.05 & 0.05 & 0 & 0.6 \\ 0 & 0.4 & 0.1 & 0.5 & 0 \\ 0 & 0.3 & 0 & 0 & 0.7 \end{bmatrix}$$

upon reordering:

$$\{1, 3, 4, 2, 5\}$$

$$P = \begin{bmatrix} * & * & * & * & * \\ - & - & - & - & - \\ 0 & 1 & * & & \end{bmatrix}$$

Block Upper Triangular



← don't listen to people outside.

$$P_{(2,5)} = \begin{bmatrix} 0.55 & 0.45 \\ 0.3 & 0.7 \end{bmatrix}$$

$$\omega^T (P - I) = 0 \rightarrow \omega = \frac{[2 \ 3]}{5}$$

$$\omega_{\text{total}} = [0 \ 0.4 \ 0 \ 0 \ 0.6]$$

Thm: Consider $x(k+1) = P x(k)$, P is row stochastic ($\lambda_0 = 1$)

Then $|\lambda_i| < 1 \ \forall \ i > 0$ (& therefore consensus is achieved) iff

G contains a "globally reachable node" (i.e. rooted directed in-branching)

and the **subgraph** of **globally reachable** nodes is **"aperiodic"**.

(submatrix of P is primitive)

Let P be row stochastic $x(k+1) = P x(k)$

Fact: $\min_i x_i(0) \leq \min_i x_i(k) \leq \min_i x_i(k+1) \leq \max_i x_i(k+1) \leq \max_i x_i(k) \leq \max_i x_i(0)$

$$\min_i x_i(k+1) = \sum_{h=1}^n p_{ih} x_h(k)$$

$$\geq \sum_{h=1}^n p_{ih} \left(\min_j x_j(k) \right) = \min_j x_j(k)$$

Let $V(x) = \max_i x_i - \min_i x_i$

Fact: $V(Ax) = V(x)$

$$\left. \begin{array}{l} \max_i (Ax)_i \leq \max_i x_i \\ -\min_i x_i \geq -\min_i (Ax)_i \end{array} \right\}$$

$$\max_i x_i - \min_i x_i \geq \max_i (Ax)_i - \min_i (Ax)_i$$

$$V(x) \geq V(Ax)$$

Does $V(x(k))$ always strictly decrease? No, $P=I$.

Even for systems that achieve consensus, V may not always strictly decrease.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$



$$1 \rightarrow 1 \rightarrow 1/2$$

$$x(0) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad x(1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad x(2) = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$$

Definition:

1-coefficient of ergodicity:

$$\tau_1(P) = \max_{\substack{\|y\|_1=1 \\ y \perp \mathbf{1}}} \|y^T A\|_1 = \frac{1}{2} \max_{i,j} \sum_{h=1}^n |a_{ih} - a_{jh}|$$

$$= 1 - \min_{i,j} \sum \min \{a_{ih}, a_{jh}\}$$

Prop: Given a row stochastic P :

(i) $V(Px) \leq \tau_1(P) V(x)$

(ii) $\tau_1(P) \leq 1$ iff P is scrambling = any pair of nodes have a common out-neighbor

Consider $x^+ = P^2 x$

$$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$

now it is scrambling !!

$$\tau_1(P^2) = \max \begin{cases} (1,2) & 0 \\ (1,3) & 0.5 \\ (2,3) & 0.5 \end{cases} = 0.5$$