Prismatic & Revolute joints will be studied.

X, Y : set of elements

Cartesian Product : Ordered pairs (x, y) $\quad x \in X,\ y \in Y$

$$X \times Y = \{ (x, y) \mid x \in X,\ y \in Y \}$$

Binary Operation $\quad \phi : \underbrace{S}_{\text{some set}} \times S \to S$

Injection : If $f(x) = f(y) \Rightarrow x = y$

Subjection : Every element has a pre-image.

Bijection : Both injection & bijection.

Group : Set & Binary operation [associative, identity, inverse, closure]

- $\phi(a, \phi(b, c)) = \phi(\phi(a, b), c)$
- $\exists$ an identity element, $e \in G$: $ae = ea = a$
- $\forall a \in G,\ \exists a^{-1} \in G$ s.t. $a \cdot a^{-1} = a^{-1} \cdot a = e$
- $\forall g_1, g_2 \in G$, $g_1 \cdot g_2 \in G$

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi > 0$$

We can prove uniqueness of identity & inverse using contradiction.

$S = \{a, b, c\}$ $\quad G$ be the set of all one-to-one maps of S onto S.

$$G = \{e, f, g, f^2, fg, gf\} \leftarrow \text{group under composition.}$$

Homomorphism : Mapping from G to $\bar{G}$ is homomorphism if $\forall\ a, b \in G$, $\phi : G \to \bar{G}$

$$\phi(a \cdot b) = \phi(a) * \phi(b)$$

Ex: $\phi(x) = a^x$

$$\phi(x + y) = \phi(x) \cdot \phi(y)$$

$$\phi : \underset{+}{\mathbb{R}} \to \underset{\cdot}{\mathbb{R}^+}$$

Ex: 2x2 Real matrices with det $\neq 0$.

$\phi = \det : \bigcirc \to \mathbb{R}$ ← under multiplication

$\phi(A) \cdot \phi(B) = \phi(AB)$

Inner product:

1. $\langle x, y+z \rangle = \langle x,y \rangle + \langle x,z \rangle$
2. $\langle x+y, z \rangle = \langle x, z \rangle + \langle y,z \rangle$
3. $a \langle x, y \rangle = \langle ax, y \rangle$
4. $\langle x, ay \rangle = \langle x, y \rangle \bar{a}$
5. $\overline{\langle x, y \rangle} = \langle y,x \rangle$
6. $\langle x, x \rangle$ = always real ≥ 0

   $\langle x, x \rangle = 0$ iff $x = 0$
7. $e_{\{1,\dots,n\}}$ is a standard basis of K^n

   then $e_i = (0, 0, \dots 0, \underset{i^{th}}{1}, 0 \dots 0)$

   $$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$
8. Inner product is non degenerate

   if $\langle x,y \rangle = 0 \ \forall y$, then $x = 0$

Definition: length $|x|$ of $x \in K^n$ is $|x| = \sqrt{\langle x,x \rangle}$

Prop: if $x, y \in K^n$ & $A \in M_n(R)$ then $\langle xA, y \rangle = \langle x, y\bar{A}^T \rangle$

(inner product is defined as $\langle x,y \rangle = x\bar{y}$)

(Quaternions)
$K \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$

$$xA = [x_1 \ x_2 \ x_3 \ \cdots \ x_n] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

$$= (x_1 a_{11} + x_2 a_{21} \cdots + x_n a_{n1}, \ x_1 a_{12} + \cdots + x_n a_{n2}, \ \cdots, \cdots x_n a_{nn})$$

$$y\bar{A}^T = (y_1 \bar{a}_{11} + y_2 \bar{a}_{12} + \cdots + y_n \bar{a}_{1n}, \ \cdots, \ y_n \bar{a}_{nn})$$

why there, we show that $\langle xA, y \rangle = \langle x, y\bar{A}^T \rangle$

$\overline{\alpha+\beta} = \bar{\alpha} + \bar{\beta}$ $\quad \overline{\alpha\beta} = \bar{\beta}\,\bar{\alpha}$

(we use the fact that conjugate commutes with sum & reverses product)

Orthogonal group: $O(n,K) = \{ A \in M_n(K) \mid \langle xA, yA \rangle = \langle x,y \rangle \ \forall \ x,y \in K^n \}$

(Quaternion product is faster and efficient, than rotational matrices)

* (There's no group over $\mathbb{R}^3$) *

To prove: $\Theta(n, K)$ is a group under matrix multiplication.

① Closure ✓ $(XA = Z,\ YA = W$, then $A, B \in \Theta$, $\langle ZB, WB \rangle = \langle Z, W \rangle = \langle X, Y \rangle)$

$X A \bar{A}^T Y^T = X Y^T$

$X B \bar{B}^T Y^T = X Y^T$

$AB\, \bar{B}^T \bar{A}^T = I$

② Identity ✓

③ Inverse ✓

$A \bar{A}^T = I$

$\bar{A}^T \overline{(\bar{A}^T)}^T = I$

$A^{-1} = \bar{A}^T$

④ Associative ✓

$\langle e_i A, e_j A \rangle = \delta_{ij}$; $e_i A = i^{th}$ row of A

$= (i, j)^{th}$ entry of $A \bar{A}^T$

$A \bar{A}^T = I$

↳ take bar transpose: $\overline{(A \bar{A}^T)}^T = \bar{A}^T A = I$

↳ Therefore it is both the left & right inverse.

Altn for inverse: $\langle x, y \rangle = \langle x A^{-1} A, y A^{-1} A \rangle$

$= \langle x A^{-1}, y A^{-1} \rangle$

For $k = \mathbb{R}$, $\Theta(n, K)$ is called orthogonal group

$k = \mathbb{C}$ unitary

$k = \mathbb{H}$ symplectic

Proposition: let $A \in \Theta(n, K)$ then

1. $\langle e_i A, e_j A \rangle = \delta_{ij}$
2. Columns of A forms an orthonormal basis
3. Rows of A forms an orthonormal basis

Proposition: If $A \in GL(n, \mathbb{R})$ preserves length, then $A \in \Theta_n$ (over Reals)

Proof: $\langle (x+y)A, (x+y)A \rangle = \langle (x+y), (x+y) \rangle$

$\langle xA, yA \rangle + \langle yA, xA \rangle = \langle x, y \rangle + \langle y, x \rangle$

$\Rightarrow$ {using the fact it is symmetrical over Reals}

$\Rightarrow \langle xA, yA \rangle = \langle x, y \rangle$

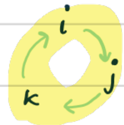
$SO(n, \mathbb{R})$ - Special Orthogonal Group

From orthogonality, we know $\det(A) \cdot \overline{\det(A)} = 1$

$\Rightarrow \|\det(A)\|^2 = 1$

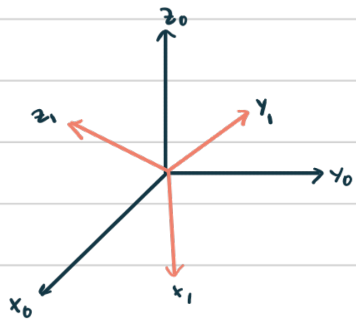
For $SO(n, \mathbb{R})$: $\det(A) = 1$

Quaternion product



	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

$\int e^{i\theta}$ $\int e^{i\theta}$



$$V_0 = \underbrace{R_1^0}_{\text{Rotation Matrix}} V_1$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}$$

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

$$\hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}_{3\times 3}$$

$$\underbrace{\vec{a} \times \vec{b}}_{\text{Vector cross product}} = \underbrace{\hat{a}}_{\text{matrix}} \cdot \underbrace{b}_{\text{vector}}$$

$\omega \in \mathbb{R}^3$ is the angular velocity of a rigid body about an axis and ω has unit length.

$$\dot{q} = \hat{\omega} q \longrightarrow q(t) = e^{\hat{\omega} t} q_0$$

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

(ij^{th} max term)

(max term) ① ⓜ $\left(\frac{nm^2}{2!}\right)$ $\left(\frac{n^2m^3}{3!}\right)$ $\left(\frac{n^{k-1}m^k}{k!}\right)$ $\frac{A^k}{k!}$

I	$A/1!$	$A^2/2!$	$A^3/3!$	$A^k/k!$
1^{st}	2^{nd}	. . . . . .		$(k+1)^{th}$

k^{th} : $\frac{n^{k-2} m^{k-1}}{(k-1)!}$

Ratio test : $\limsup_{k\to\infty} \frac{a_{(k+1)^{th}}}{a_{k^{th}}} = \limsup_{k\to\infty} \frac{nm}{k} = 0$

$\Rightarrow$ Absolute convergence.

we used these

① Absolute conv $\Rightarrow$ conv

② $\sum a_n \to$ converges $\{a_n > 0\}$

then $\sum b_n \to$ converges if $\{a_n > b_n > 0\}$

if $AB = BA$ then $e^A \cdot e^B = e^{A+B}$

$A \in \mathbb{R}^{n \times n}$ then e^A is non singular.

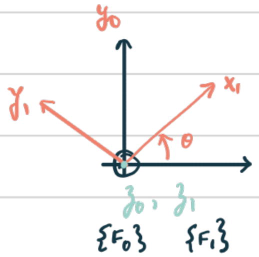
Proof: $I = e^0 = e^{A-A} = e^A \cdot \underbrace{e^{-A}} = e^A \cdot (e^A)^{(-1)} \Rightarrow$ non singular matrix

if $A + A^T = 0$, then $\underbrace{(e^A)}_{\text{Orthogonal matrix}} \cdot (e^A)^T = I$

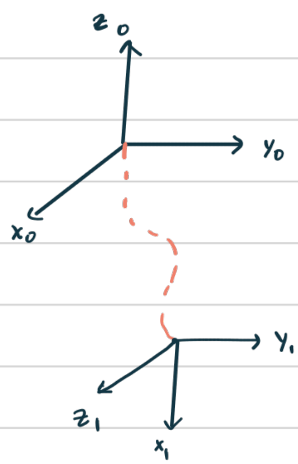
$e^A \to$ orthogonal $\not\Rightarrow$ $A + A^T = 0$

$\to e^{ABA^{-1}} = A e^B A^{-1}$

$\to A = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} \qquad e^A = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$



CCW $\to$ Rotations are true.



$Rot(z_0, \theta) = Rot(x_1, -\theta)$

$$R_1^0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$A = \{F^0\} \qquad B = \{F^1\}$

Similarity transform

$$B = (R_1^0)^{-1} A R_1^0$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix} = Rot(x, -\theta)$$

Representation of Rotation Matrices

1. Euler angle representation
2. Axis angle representation
3. Quaternion representation

$\hat{a} \in so(3)$ $\quad$ $\hat{a}^2 = aa^T - \|a\|^2 I$

$\hat{a}^3 = -\|a\|^2 \hat{a}$

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\theta + \frac{(\hat{\omega}\theta)^2}{2!} + \dots = I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots\right)\hat{\omega}$$

$\|\omega\| = 1$

$$+ \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \dots\right)\hat{\omega}^2$$

{Euler - Lexell formula} $\quad e^{\hat{\omega}\theta} = I + \sin\theta\, \hat{\omega} + (1 - \cos\theta)\, \hat{\omega}^2$

{Rodrigues formula}

$\|\omega\| \neq 1$, then $\quad e^{\hat{\omega}\theta} = I + \frac{\hat{\omega}}{\|\omega\|} \sin(\|\omega\|\theta) + \frac{\hat{\omega}^2}{\|\omega\|^2}\left(1 - \cos(\|\omega\|\theta)\right)$

$$R = e^{\hat{\omega}\theta} = \boxed{\begin{bmatrix} \omega_1^2 v_\theta + c_\theta & \omega_1\omega_2 v_\theta - \omega_3 s_\theta & \omega_1\omega_3 v_\theta + \omega_2 s_\theta \\ \omega_1\omega_2 v_\theta + \omega_3 s_\theta & \omega_2^2 v_\theta + c_\theta & \omega_2\omega_3 v_\theta - \omega_1 s_\theta \\ \omega_1\omega_3 v_\theta - \omega_2 s_\theta & \omega_2\omega_3 v_\theta + \omega_1 s_\theta & \omega_3^2 v_\theta + c_\theta \end{bmatrix}}$$

$v_\theta = 1 - \cos\theta$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \cdot \frac{1}{2\sin\theta}$$

$$\text{Trace}(R) = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$$

sum of eigen values.

$$\theta = \pm \cos^{-1}\left[\frac{\text{Trace}(R) - 1}{2}\right]$$

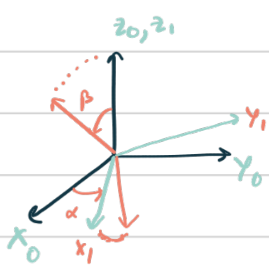
How many rotational axes. Euler angles.

takes vectors in 0

$R_1^0 = R_{01}$

gives vectors in 1

Euler Angle: $\quad$ z x z
$\quad \alpha \; \beta \; \gamma$



$R_3^0 = R_z(\alpha)\, R_y(\beta)\, R_z(\gamma)$

Rotations about the current frame - post multiply.
the fixed frame - pre multiply

$$R = \begin{bmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha c_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta \\ s_\alpha c_\beta c_\gamma + c_\alpha s_\gamma & -s_\alpha c_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta \\ -s_\beta c_\gamma & s_\beta s_\gamma & c_\beta \end{bmatrix}$$

{we assume $\sin\beta > 0$}

$$\beta = \pm \cos^{-1}(R_{33})$$
$$\gamma = \operatorname{atan2}(R_{32}, -R_{31}) \quad \{\sin\beta \neq 0\}$$
$$\alpha = \operatorname{atan2}(R_{23}, R_{13})$$

↪ Issue of multiplicity of solutions.

$e^{i\theta} = \cos\theta + i\sin\theta$

$(x+iy)$ $\qquad e^{i\theta}(x+iy) = (x\cos\theta - y\sin\theta) + i(y\cos\theta + x\sin\theta)$

$$\begin{bmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad R v \triangleq h v h^* \quad \underset{\text{(unit quaternion)}}{}$$

$$Q = (q_0, \vec{q}) = \overset{(q_0,\vec{q})}{q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k}} \qquad Q^* = (q_0, -\vec{q})$$
$$P = (p_0, \vec{p})$$

$$QP = \boxed{(p_0 q_0 - \vec{q}\cdot\vec{p},\ q_0\vec{p} + p_0\vec{q} + \vec{q}\times\vec{p})}$$
$$QQ^* = (1, \vec{0})$$

$$Rv \leftrightarrow Q v Q^* \qquad v = (0, \vec{v})$$

$$Qv = (\underbrace{-\vec{q}\cdot\vec{v}}_{x_0}, \underbrace{q_0\vec{v} + \vec{q}\times\vec{v}}_{x}) = X = (x_0, \vec{x})$$

$$XQ^* = (x_0 q_0 + \vec{x}\cdot\vec{q}, -x_0\vec{q} + q_0\vec{x} - \vec{x}\times\vec{q})$$

$$\Rightarrow QvQ^* = (-(q\cdot v)q_0 + (q_0\vec{v} + \vec{q}\times\vec{v})\cdot\vec{q},\ (\vec{q}\cdot\vec{v})\vec{q} + q_0(q_0\vec{v} + \vec{q}\times\vec{v}) - (q_0\vec{v} + \vec{q}\times\vec{v})\times\vec{q})$$

$$= (0,\ (\vec{q}\cdot\vec{v})\vec{q} + q_0^2\vec{v} + 2q_0\vec{q}\times\vec{v} + (\vec{q}\cdot\vec{v})\vec{q} - (\vec{q}\cdot\vec{q})\vec{v})$$

$$M = \begin{bmatrix} 1-2(q_2^2+q_3^2) & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_0q_3 + q_1q_2) & 1-2(q_1^2+q_3^2) & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_0q_1 + q_2q_3) & 1-2(q_2^2+q_1^2) \end{bmatrix}$$

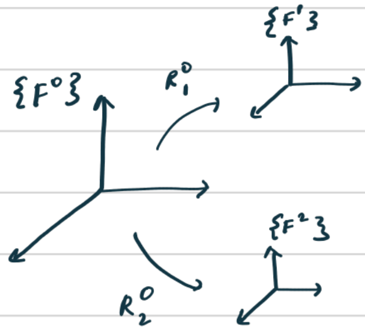
$$|q_0| = \sqrt{\frac{\mathrm{Trace}(R)+1}{4}}$$

$$R_z \triangleq (\cos \phi/2,\ \sin \phi/2\ \hat{k})$$

$$|q_1| = \sqrt{\frac{2r_{11} - \mathrm{Trace}(R)+1}{4}}$$

$$R = e^{\hat{\omega}\theta}$$

$$Q = (\cos \theta/2,\ \vec{\omega} \sin \theta/2)$$



$$R_2^0 = R_1^0 R_2^1$$

$$R_2^1 = (R_1^0)^{-1} R_2^0$$

$$q_1 v q_1^* = v_1 \quad \Rightarrow \quad v = q_1^* v_1 q_1$$

$$q_2 v q_2^* = v_2$$

$$\Rightarrow \quad v_2 = q_2 q_1^* v_1 (q_2 q_1^*)^*$$

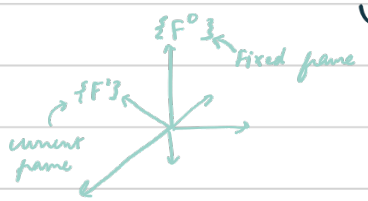
$$R_2^1 \triangleq q_2 q_1^*$$

Example

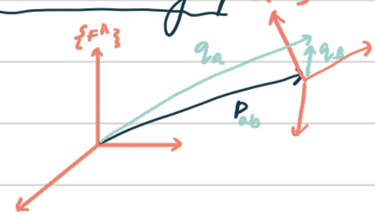
Workout:

$$R_x(\pi/2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad R_x \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\rightarrow$ compute q $\rightarrow$ compute $q v q^*$



Special Euclidean Group



General Rigid body transformation

$$\vec{q}_a = \vec{p}_{ab} + R_{ab} \vec{q}_b$$

Displacements: cannot be represented by $n \times n$ matrix.

$$H_n : \ x_{n+1} = 1$$

$$\mathbb{R}^n \longrightarrow (x_1, x_2, \ldots, x_n)$$

Vector $\longrightarrow (x_1, x_2, \dots, x_n, 1)$

$$T = [A, d] \longrightarrow \left[\begin{array}{c|c} A & d \\ \hline 0 & 1 \end{array}\right]$$

Homogeneous Representation: Refers to the fact that the coordinate vector $(\vec{x}, 1)$ is the homogeneous coordinate.

Transformations of points and vectors by rigid transformation in terms of matrices & vectors in $\mathbb{R}^4$.

$$\vec{q} = (q_x, q_y, q_z)$$

$$(\vec{q}, 1) = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} \qquad \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{vectors} : \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix}$$

$$\bar{q}_a = g_{ab} \bar{q}_b$$

$$\begin{bmatrix} q_a \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}}_{\in SE(3)} \begin{bmatrix} q_b \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab} q_b + p_{ab} \\ 1 \end{bmatrix}$$

$$SE(3) = \{ (p, R) \mid p \in \mathbb{R}^3, R \in SO(3) \}$$
$$= \mathbb{R}^3 \rtimes SO(3)$$

semi direct product.

Lemma: SE(3) is a group. $\left\{ \begin{array}{ll} \to \text{Closure} & \to \text{Identity} \\ \to \text{Associativity} & \to \text{Inverse} \end{array} \right\}$

$$g^{-1} = \left[\begin{array}{c|c} R^T & -R^T p \\ \hline 0 & 1 \end{array}\right]$$

$$\bar{R}_x = \left[\begin{array}{c|c} R_x & 0 \\ \hline 0 & 1 \end{array}\right] \qquad T_x = \left[\begin{array}{c|c} I & \begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \\ \hline 0 & 1 \end{array}\right] \qquad \begin{bmatrix} R_x & 0 \\ 0 & 1 \end{bmatrix}$$

$\left\{ \begin{array}{c} R_y \\ R_z \end{array} \right\}$

$$\bar{R}_x T_x = \left[\begin{array}{c|c} R_x & R_x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \hline 0 & 1 \end{array}\right]$$

$$\neq$$

$$T_x \bar{R}_x = \left[\begin{array}{c|c} R_x & \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \hline 0 & 1 \end{array}\right]$$

$$T_x T_y = T_y T_x$$

Elements of SE(3) represent rigid motions

$$\| g q - g p \| = \| q - p \|$$

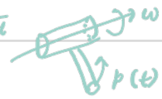
$$\longrightarrow g_* (v \times w) = (g_* v) \times (g_* w)$$

Finding the exponential map.

$\vec{\omega} \in \mathbb{R}^3$

$\|\vec{\omega}\| = 1$

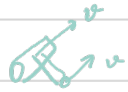
Revolute joint



q is a point on axis of rotation.

$$\dot{p}(t) = \vec{\omega} \times (p(t) - q)$$
$$= \vec{\omega} \times \vec{p}(t) - \vec{\omega} \times \vec{q}$$

Prismatic joint



$\dot{p}(t) = v$

$\swarrow$ skew

SO(3) $\leftarrow$ **so(3)**

ω, θ are the exponential coordinates $e^{\hat{\omega}\theta}$

$\hat{\omega}$: skew symmetric matrix of ω

SE(3)

(v, ω) are twist coordinates of $\hat{\xi}$, $e^{\hat{\xi} t}$

where $\hat{\xi} = \left[\begin{array}{c|c} \hat{\omega} & v \\ \hline 0 & 0 \end{array}\right] \in se(3)$

$e^{\hat{\xi}} \in SE(3)$

* q is a point on axis of rotation.

$$\begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix} = \left[\begin{array}{c|c} \hat{\omega} & -\vec{\omega} \times \vec{q} \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} \vec{p} \\ 1 \end{bmatrix}$$

vee operation : $\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}^{\vee} = \begin{pmatrix} v \\ \omega \end{pmatrix}$

4×4 $\qquad$ 6×1

Wedge operation : $\begin{pmatrix} v \\ \omega \end{pmatrix}^{\wedge} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$

6×1

Proposition : Exponential map from $se(3)$ to $SE(3)$

Given $\hat{\xi} \in se(3)$ & $\theta \in \mathbb{R}$, the exponential map

$$e^{\hat{\xi}\theta} \in SE(3)$$

$$e^{\hat{\xi}\theta} = I + \hat{\xi}\theta + \frac{\hat{\xi}^2 \theta^2}{2!} + \dots$$

Case 1: $\omega = 0 \Rightarrow \hat{\xi}^2 = 0 \Rightarrow \hat{\xi}^n = 0$

$$e^{\hat{\xi}\theta} = I + \hat{\xi}\theta = \left[\begin{array}{c|c} I & v\theta \\ \hline 0 & 1 \end{array}\right] \in SE(3)$$

Case 2: $\omega \neq 0$, WLOG : $\|\omega\|_2 = 1$

$$\hat{\xi}^2 = \left[\begin{array}{c|c} \hat{\omega}^2 & \hat{\omega} v \\ \hline 0 & 0 \end{array}\right] \qquad \hat{\xi}^3 = \left[\begin{array}{c|c} \hat{\omega}^3 & \hat{\omega}^2 v \\ \hline 0 & 0 \end{array}\right] \qquad \hat{\xi}^n = \left[\begin{array}{c|c} \hat{\omega}^n & \hat{\omega}^{n-1} v \\ \hline 0 & 0 \end{array}\right]$$

$$e^{\hat{\xi}\theta} = I + \sum_{i=1}^{\infty} \left[\begin{array}{c|c} \hat{\omega}^i & \hat{\omega}^{i-1} v \\ \hline 0 & 0 \end{array}\right] \frac{\theta^i}{i!}$$

$$= \left[\begin{array}{c|c} e^{\hat{\omega}\theta} & \sum_{i=1}^{\infty} \frac{\hat{\omega}^{i-1}\theta^i}{i!} v \\ \hline 0 & 1 \end{array}\right]$$

$\hat{a} \in so(3)$ $\hat{a}^2 = aa^T - \|a\|^2 I$

$\hat{a}^3 = -\|a\|^2 \hat{a}$

$$\theta \left(I + \frac{\hat{\omega}\theta}{2!} + \frac{\hat{\omega}^2\theta^2}{3!} + \frac{\hat{\omega}^3\theta^3}{4!} + \cdots \right) v$$

$\cancel{I\theta} +$ $\frac{\hat{\omega}\theta^2}{2!} + \frac{\hat{\omega}^2\theta^3}{3!}$

$\boxed{\text{Ans} = (I - e^{\hat{\omega}\theta})\hat{\omega}v + (\hat{\omega}^2 + I)v\theta}$

$- \frac{\hat{\omega}\theta^4}{4!} - \frac{\hat{\omega}^2\theta^5}{5!}$

$\theta + \hat{\omega}^2(\theta - \sin\theta)$

$\hat{\omega}(1 - \cos\theta)$

$$e^{\hat{\omega}\theta} = I + \sin\theta\, \hat{\omega} + (1-\cos\theta)\hat{\omega}^2$$

$+ \frac{\hat{\omega}\theta^6}{6!} - \frac{\hat{\omega}^6\theta^7}{7!}$

$$\theta \left(I + \hat{\omega}\left(\frac{1-\cos\theta}{\theta}\right) + \frac{\hat{\omega}^2}{\theta}\left(\theta - \sin\theta\right)\right) v$$

$$= \left(\theta + \hat{\omega}(1-\cos\theta) + \hat{\omega}^2(\theta - \sin\theta)\right) v$$

$$\text{Ans} = \left(\left(-\sin\theta\, \hat{\omega}^2 + (1-\cos\theta)\hat{\omega}\right) v + (\hat{\omega}^2 + I) v\theta \right)$$

$$\left(\theta + (\theta - \sin\theta)\hat{\omega}^2 + (1-\cos\theta)\hat{\omega}\right) v$$

$\Rightarrow e^{\hat{\xi}\theta} \in SE(3)$, $g_{ab}(\theta) = e^{\hat{\xi}\theta} g_{ab}(0)$

Surjectivity of Exponential map

$g \in SE(3)$, $\exists\ \hat{\xi} \in se(3)$ and $\theta \in \mathbb{R}$

s.t. $g = e^{\hat{\xi}\theta}$

Case 1: $R = I$

$$\hat{\xi} = \left[\begin{array}{c|c} 0 & \frac{p}{\|p\|} \\ \hline 0 & 0 \end{array}\right] \qquad \theta = \|p\|$$

Case 2: $R \neq I$

$$e^{\hat{\xi}\theta} = \left[\begin{array}{c|c} e^{\hat{\omega}\theta} & \boxed{(I - e^{\hat{\omega}\theta})\hat{\omega}v + (\hat{\omega}^2 + I)v\theta} \\ \hline 0 & 1 \end{array}\right]$$

Need to show $[\theta + \hat{\omega}(1-\cos\theta) + \omega^2(\theta - \sin\theta)]$ is full rank

$(\omega\omega^T - I)(\theta - \sin\theta)$

$$\cancel{\theta} + \hat{\omega}(1-\cos\theta) + \omega\omega^T\theta - \omega\omega^T\sin\theta - \cancel{\theta} + \sin\theta$$

$$\left(\theta + \hat{\omega}(1-\cos\theta) + \hat{\omega}^2(\theta - \sin\theta)\right) v = 0$$

$$\Big(\underbrace{\theta v}_{0} + \underline{\underline{\vec{\omega}\times\vec{v}}}\,(1-\cos\theta) + \omega\times(\omega\times v)(\theta - \sin\theta)\Big) = 0$$

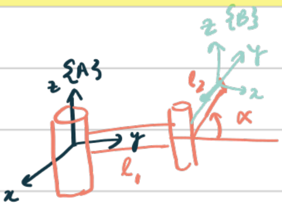
$\cdot\,\vec{v}$

$\cdot\,(\vec{\omega}\times\vec{v})$ $\qquad = 0$

$$\|\omega\times v\|(1-\cos\theta) = 0$$

$\theta = 0,\ 2\pi$

$\omega = \lambda v$



$$g_{ab}(\alpha) = \left[\begin{array}{ccc|c} c_\alpha & -s_\alpha & 0 & -l_2\sin\alpha \\ s_\alpha & c_\alpha & 0 & l_2\cos\alpha + l_1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[(I - e^{\hat{\omega}\alpha})\hat{\omega} + \omega\omega^T\alpha\right] v = p_{ab} \qquad \omega = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$(I - e^{\hat{\omega}\alpha})\hat{\omega} = \begin{bmatrix} s_\alpha & c_\alpha - 1 & 0 \\ 1-c_\alpha & s_\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\omega\omega^T\alpha = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

$$\begin{bmatrix} s_\alpha & c_\alpha - 1 & 0 \\ 1-c_\alpha & s_\alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} -l_2 s_\alpha \\ l_1 + l_2 c_\alpha \\ 0 \end{bmatrix}$$

$v_z = 0$ $\qquad \alpha\left[s_\alpha^2 + (c_\alpha - 1)^2\right]$

$\alpha(2 - 2c_\alpha)$

s_α $\qquad [\,s_\alpha v_x + (c_\alpha - 1) v_y = -l_2 s_\alpha$

$(1-c_\alpha)$ $\qquad (1-c_\alpha) v_x + s_\alpha v_y = l_1 + l_2 c_\alpha$

$$((c_\alpha - 1)^2 + s_\alpha^2) v_y = \cancel{-l_2 s_\alpha c_\alpha} + l_2 s_\alpha + l_1 s_\alpha + \cancel{l_2 c_\alpha s_\alpha}$$

$$v_y = \frac{(l_1 + l_2) s_\alpha}{2 - 2\cos\alpha} = \frac{(l_1+l_2)\, 2 s_{\alpha/2} c_{\alpha/2}}{2 \cdot 2 \cdot s^2_{\alpha/2}}$$

$$= \frac{(l_1+l_2)}{2}\cot\alpha/2$$

$$v_x = \frac{-l_2 + l_1 + l_2 c_\alpha - l_1 c_\alpha}{2 - 2\cos\alpha}$$

$$= \frac{l_1(1-c_\alpha) + l_2(c_\alpha - 1)}{2 - 2c_\alpha} \Rightarrow \left\{ v_x = \frac{l_1 - l_2}{2} \right\}$$

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} A_{ij} \end{bmatrix}^T \Rightarrow \xi = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} (l_1 - l_2)/2 \\ \dfrac{(l_1+l_2) s_\alpha}{2(1-c_\alpha)} \\ 0 \\ \hdashline 0 \\ 0 \\ 1 \end{bmatrix}$$

cofactors of a_{ij}

$g_{ab} = e^{\hat{\xi}\alpha} g_{ab}(0)$ $\qquad$ $g_{ab}(0) = \left[\begin{array}{c|c} I & \begin{smallmatrix}0\\ l_1+l_2\\ 0\end{smallmatrix} \\ \hline 0 & 1 \end{array}\right]$

$g(\alpha) = e^{\hat{\xi}\alpha} = g_{ab}\, g^{-1}_{ab}(0)$

$$= \left[\begin{array}{ccc|c} c\alpha & -s\alpha & 0 & l_1 s\alpha \\ s\alpha & c\alpha & 0 & l_1(1-c\alpha) \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$\omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\qquad$ $-\omega \times q$ $\qquad$ $\frac{s_\alpha}{2(1-c_\alpha)} l_1 s_\alpha + \frac{1}{2} l_1(1-c_\alpha)$

$v = \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}$ $\qquad$ $-\frac{1}{2} l_1 s_\alpha + \frac{s_\alpha \cancel{l_1(1-c_\alpha)}}{\cancel{2(1-c_\alpha)}}$

l_1

$\boxed{v = -\omega \times q}$ $\qquad$ $\xi = \begin{bmatrix} l_1 \\ 0 \\ 0 \\ \hdashline 0 \\ 0 \\ 1 \end{bmatrix}$

point on the axis $= \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}$

Screws :

Twist is normally associated with a screw motion

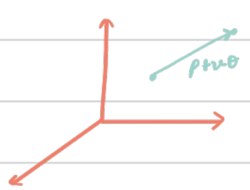
Pitch : $h = \frac{d}{\theta}$ $\quad$ (m /radian)

Net translation after rotating by θ^c is $h\theta^c$

If $q \in \{\text{Axis of rotation}\}$ & $\omega \in \mathbb{R}^3$

$l = \{ q + \lambda\omega : \lambda \in \mathbb{R} \}$

For 0 rotation the axis defined as the line through the origin in the direction of v $\quad$ ($|v|=1$)



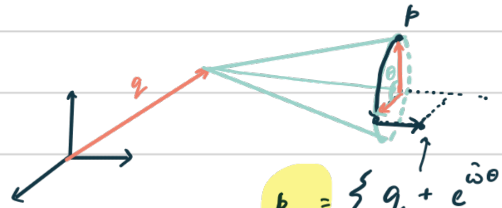
infinite pitch

Magnitude is the amount of translation

Definition of screw motion: Screw consists of an axis l, pitch h and magnitude M

Rotation : $M = \theta$

If $h = \infty$, then screw is pure translation.



$$p_{new} = \{ q + e^{\hat{\omega}\theta}(\vec{p} - \vec{q}) + h\theta\vec{\omega} \}$$

$$g = \left[\begin{array}{c|c} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ \hline 0 & 1 \end{array}\right] \triangleq \left[\begin{array}{c|c} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})\hat{\omega}v + \omega\omega^T v\theta \\ \hline 0 & 1 \end{array}\right]$$

$$v = -\omega \times q + h\omega \qquad \xi = (v, \omega)$$

Screw coordinates of a twist:

1. Pitch: $h = \dfrac{\omega^T v}{\|\omega\|^2}$, if $\omega = 0 \Rightarrow h \to \infty$

2. Axis: $l = \begin{cases} \dfrac{\omega \times v}{\|\omega\|^2} + \lambda\omega & : \lambda \in \mathbb{R},\ \omega \neq 0 \\ 0 + \lambda v & : \lambda \in \mathbb{R},\ \text{if } \omega = 0 \end{cases}$

3. Magnitude: $M = \begin{cases} \|\omega\| & \text{if } \omega \neq 0 \\ \|v\| & \text{if } \omega = 0 \end{cases}$

If we choose unit magnitude ($\|v\| = 1$ for $\omega = 0$) then $\hat{\xi}\theta$ has magnitude $M = \theta$

Quiz 2

Chasles Theorem

Every rigid body motion can be realized by a rotation about an axis in space and translation parallel to that axis

$p(0) \in \mathbb{R}^3$

$p(\theta) = e^{\hat{\xi}\theta} p(0)$

Screw motions correspond to twists.

Theorem: Given a screw with axis l, pitch h and magnitude M then there exists a unit magnitude twist ξ s.t. the rigid motion associated with the screw is generated by the twist $M\xi$.

Velocity of a rigid body:

$R : \mathbb{R} \longrightarrow SO(3)$

Then $\dot{R}(t) \notin SO(3)$

$\notin so(3)$

A: Spatial frame

B: Body frame

$q_a(t) = R_{ab}(t)\, q_b$

$\dot{q}_a(t) = \dot{R}_{ab}(t)\, q_b = \underbrace{\dot{R}_{ab} R_{ab}^{-1}}_{=\Omega} q_a(t)$

$\dot{R} R^T = \Omega \in so(3)$

$\dot{R}R^T + \dot{R}^T R = 0$

$\Rightarrow (\dot{R}R^T) + (\dot{R}R^T)^T = 0$

Instantaneous spatial angular velocity : $\omega^s_{ab} \in \mathbb{R}^3 \to \hat{\omega}^s_{ab} := \dot{R}_{ab} R^T_{ab}$

" body " : $\omega^b_{ab} \in \mathbb{R}^3 \to \hat{\omega}^b_{ab} := R^T_{ab} \dot{R}_{ab}$

$\hat{\omega}^b_{ab} = R^T \hat{\omega}^s_{ab} R$

$\omega^b_{ab} = R^T_{ab} \omega^s_{ab}$

Example: $R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix}$

$\dot{R}_x(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -s_\theta & -c_\theta \\ 0 & c_\theta & -s_\theta \end{bmatrix}$

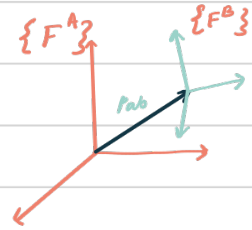
$\dot{R}_x(\theta) R^T_x(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \dot{\theta} \Rightarrow \omega^s_x(\theta) = \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}$

$R^T_x(\theta)\, \dot{R}_x(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \dot{\theta} \Rightarrow \omega^b_x(\theta) = \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}$

$g_{ab} \in SE(3)$

$\underset{\mathbb{R}}{\underset{\uparrow}{t}} \rightarrow \underset{SE(3)}{\underset{\uparrow}{g_{ab}(t)}} = \left[\begin{array}{c|c} R(t) & p(t) \\ \hline 0\,0\,0 & 1 \end{array}\right]$

$$\dot{g}_{ab}(t) = \begin{bmatrix} \dot{R}(t) & \dot{p}(t) \\ 000 & 0 \end{bmatrix}$$



$$g^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

$$\dot{g}g^{-1} = \begin{bmatrix} \dot{R}R^T & -\dot{R}R^T p + \dot{p} \\ 0 & 0 \end{bmatrix} = \hat{V}^s_{ab}$$

$$g^{-1}\dot{g} = \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0 & 0 \end{bmatrix}$$

$$\boxed{V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix} = \begin{bmatrix} -\dot{R}R^T p_{ab} + \dot{p}_{ab} \\ (\dot{R}_{ab} R^T_{ab})^\vee \end{bmatrix}}$$

$$v_{q_a} = \hat{V}^s_{ab}\, q_a = \omega^s_{ab} \times q_a + v^s_{ab}$$

$$\hat{V}^b_{ab} = g^{-1}_{ab}\dot{g}_{ab} = \begin{bmatrix} R^T_{ab} & -R^T_{ab} p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R}_{ab} & \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R^T_{ab}\dot{R}_{ab} & R^T_{ab}\dot{p}_{ab} \\ 0 & 0 \end{bmatrix}$$

$$\boxed{V^b_{ab} = \begin{bmatrix} R^T_{ab}\dot{p}_{ab} \\ (R^T_{ab}\dot{R}_{ab})^\vee \end{bmatrix}}$$

$$v_{q_b} = g^{-1}_{ab} v_{q_a}$$
$$= g^{-1}_{ab}\dot{g}_{ab} q_b$$
$$= \hat{V}^b_{ab}(t)\, q_b$$

$$v_{q_b} = \hat{V}^b_{ab} q_b = \omega^b_{ab} \times q_b + v^s_{ab}$$

Adjoint Operation

$$V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix} \; ; \; V^b_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix}$$

$$v^s_{ab} = -\omega^s_{ab} \times p_{ab} + \dot{p}_{ab} = p_{ab} \times (R_{ab}\,\omega^s_{ab}) + R_{ab}\, v^b_{ab}$$

$$\omega^s_{ab} = R_{ab}\,\omega^b_{ab}$$

$$\boxed{\underbrace{\begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix}}_{\text{spatial}} = \underbrace{\begin{bmatrix} R_{ab} & \hat{p}_{ab}R_{ab} \\ 0 & R_{ab} \end{bmatrix}}_{\text{Adjoint Matrix}} \underbrace{\begin{bmatrix} v^b_{ab} \\ \omega^b_{ab} \end{bmatrix}}_{\text{Body}}}$$

$$Ad_g : \mathbb{R}^6 \longrightarrow \mathbb{R}^6$$

$$(Ad_g)^{-1} = (Ad_{g^{-1}}) \qquad Ad_g\, Ad_g^{-1} = I$$

$$\begin{bmatrix} R & \hat{D}R \\ 0 & R \end{bmatrix} \underbrace{\begin{bmatrix} \overset{R^T}{X} & \overset{-R^T\hat{D}}{Y} \\ Z^0 & W^{R^T} \end{bmatrix}}_{(Ad_g^{-1})} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

If $\hat{\xi} \in se(3)$ is a twist with twist coordinates ξ
then $g \in SE(3)$ $g \hat{\xi} g^{-1}$ is a twist with
twist coordinates $Ad_g \xi$

$$V^s = Ad_g V^b$$

$$g(t) = \left[\begin{array}{ccc|c} \cos\theta(t) & -\sin\theta(t) & 0 & -l_2\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) & 0 & l_1 + l_2\cos\theta(t) \\ 0 & 0 & 1 & l_0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{matrix} -s & -c & 0 \\ c & s & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$V^s = \begin{bmatrix} v^s \\ \omega^s \end{bmatrix} = \begin{bmatrix} -\dot{R}R^T p - \dot{p} \\ \dot{R}R^T \end{bmatrix} = \begin{bmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix} \end{bmatrix}$$

$$-\begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix} \times \begin{pmatrix} l_2 \sin\theta \\ l_1 + l_2\cos\theta \\ l_0 \end{pmatrix} + \begin{pmatrix} -l_2\cos\theta\,\dot{\theta} \\ l_2 \sin\theta\,\dot{\theta} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} l_1\dot{\theta} + (l_2)\cos\theta\,\dot{\theta} \\ -l_2\sin\theta\,\dot{\theta} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} l_1 \dot{\theta} \\ 0 \\ 0 \end{pmatrix} = v^s$$

$$R^T \dot{P}_{ab} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -l_2 c\dot{\theta} \\ -l_2 s\dot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} -l_2\dot{\theta} \\ 0 \\ 0 \end{bmatrix}$$

Velocity of Screw motion

F^A, F^B $$g_{ab}(\theta) = e^{\hat{\xi}\theta} g_{ab}(0)$$

Assuming a constant twist, ξ

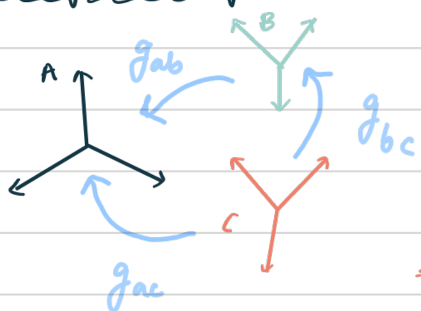
$$\frac{d}{dt}\left(e^{\hat{\xi}\theta}\right) = \hat{\xi}\dot{\theta} e^{\hat{\xi}\theta}$$

$$\hat{V}^s_{ab} = \dot{g}_{ab}(\theta)\, g^{-1}_{ab}(\theta) = \hat{\xi}\dot{\theta}$$

$$\hat{V}^b_{ab} = g^{-1}_{ab}(0)\, e^{-\hat{\xi}\theta}\, \hat{\xi}\dot{\theta}\, e^{\hat{\xi}\theta}\, g_{ab}(0)$$

$$= (Ad_{g^{-1}_{ab}(0)}\, \hat{\xi})\, \dot{\theta}$$

Coordinate transformation for velocities



$$g_{ac} = g_{ab}\, g_{bc}$$

$$\dot{g}_{ac}\, g^{-1}_{ac} = (\dot{g}_{ab}\, g_{bc} + g_{ab}\, \dot{g}_{bc})\, g^{-1}_{bc}\, g^{-1}_{ab}$$

Spatial frame

$$\Rightarrow \boxed{V^s_{ac} = V^s_{ab} + Ad_{g_{ab}} V^s_{bc}}$$

Body frame

$$\boxed{V^b_{ac} = Ad_{g^{-1}_{bc}} V^b_{ab} + V^b_{bc}}$$

Syllabus for Midterm →

Practice Problems

1. $a = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; $b = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$; $i^2 = -1$

M is a group - matrix multiplication, formed by a + b.

$b^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$ $\quad b^3 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$ $\quad b^4 = e$

$a^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$ $\quad ab^3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$ $\quad a^2b^3 = b$

$b^4 = I$ $\quad a^3b^3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$ab^3 a = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = -b$

a, b $\quad ab = a^3b^3$

$a^2 = -I$ $\quad M = \{e, a, b, a^2, a^3, b^3, ab, ab^3\}$

↳ Quaternion Group

2. A $(n \times n)$ ← n distinct eigenvalues.

$$\det(e^A) = e^{\text{trace}(A)}$$

if an eigenvalue is 0, does it work? Yes ... ☺

3) $$R = \begin{bmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha c_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta \\ s_\alpha c_\beta c_\gamma + c_\alpha s_\gamma & -s_\alpha c_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta \\ -s_\beta c_\gamma & s_\beta s_\gamma & c_\beta \end{bmatrix}$$

ω^b & ω^s = ?

Verify $\Omega^b = R^T \Omega^s R$

(α, β, γ are functions of time.)

4) Consider $SE(2)$:

$\xi_g = (v, \omega) \in \mathbb{R}^3$

$$\hat{\xi}_g = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \cos\omega & -\sin\omega & x \\ \sin\omega & \cos\omega & y \\ 0 & 0 & 1 \end{bmatrix}$$

$\hat{\omega}$

$\hat{\omega}^2 = -\omega^2 I$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} + \frac{\hat{\omega}}{2}\begin{pmatrix} v_x \\ v_y \end{pmatrix} + \frac{\hat{\omega}^2}{3!}\begin{pmatrix} v_x \\ v_y \end{pmatrix} + \frac{\omega^3}{4!}\begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$\frac{\omega}{\omega}$ $$\left(1 - \frac{\omega^2}{3!} + \frac{\omega^4}{5!} \dots\right)\begin{pmatrix} v_x \\ v_y \end{pmatrix} + \hat{\omega}\begin{pmatrix} v_x \\ v_y \end{pmatrix}\left(\frac{1}{2} - \frac{\omega^2}{4!} + \frac{\omega^4}{6!}\right)$$

$$\frac{\sin\omega}{\omega}\begin{pmatrix} v_x \\ v_y \end{pmatrix} + \frac{\hat{\omega}}{\omega^2}\begin{pmatrix} v_x \\ v_y \end{pmatrix}(1-\cos\omega)$$

$$\left\{ \frac{\sin\omega}{\omega}\begin{pmatrix} v_x \\ v_y \end{pmatrix} + \frac{1}{\omega}\begin{pmatrix} -v_y \\ v_x \end{pmatrix}(1-\cos\omega) \right\}$$

{CLB 021}

$$\dot{g} = \begin{bmatrix} -s_\theta & c_\theta & v_x \\ -c_\theta & -s_\theta & v_y \\ 0 & 0 & 0 \end{bmatrix}$$

Find the adjoint transformation for planar rigid motion that maps body velocities $\in \mathbb{R}^3$ to spatial velocities $v^s \in \mathbb{R}^3$

$$Ad_g = \left[\begin{array}{c|c} R & \begin{matrix} p_y \\ -p_x \end{matrix} \\ \hline 0\ 0 & 1 \end{array}\right]$$

$\boxed{HW-3}$

$(\widehat{Ad_g v}) = g\hat{v}g^{-1}$

$\boxed{3}$ (a) $g_{01} =$ $p \rightarrow$ what is the coordinate of 1 wrt 0 in 0.

$R \rightarrow$ of rotation by z of 0 by θ:

$$\begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $\hat{V}^s_{03} = \dot{g}_{03}\, g^{-1}_{03}$

of rotation wrt z is positive in spatial frame

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_z & 0 \\ +\omega_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) $\hat{V}^b_{03} = g^{-1}_{03}\, \dot{g}_{03}$

~~$\theta = 0$~~

$S \rightarrow$	$l_1 \dot{\theta}_2$ ✓
0
0

$l_2 \quad s_2$

$B \rightarrow \quad -l_2(\dot{\theta}_1 + \dot{\theta}_2) \quad -l_1\dot{\theta}_1$ ✓

0

0

$p_{03} = -p_{30}$ (For all time if no rotations)

$\frac{d}{dt} p_{03} = -\frac{d}{dt} p_{30}$

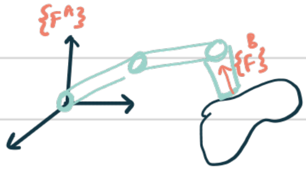
$$\left\{ g_{bc} \quad V^c_{bc} \quad g^{-1}_{bc} \right\}$$

$$Ad_{bc} \; V^c_{bc}$$

Wrenches

$$\xi = (v, \omega) \quad \hat{=} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

$$F = \begin{bmatrix} f \\ \tau \end{bmatrix} \begin{matrix} \leftarrow \in \mathbb{R}^3 & (\text{translational}) \\ \leftarrow \in \mathbb{R}^3 & (\text{rotational}) \end{matrix}$$



$g_{ab}(t)$: Homogeneous transformation matrix

V_{ab}^b : Body velocity

F_b : Applied wrench in Frame B.

$$\delta W = V_{ab}^b \cdot F_b = v \cdot f + \omega \cdot \tau$$

$$W = \int_{t_1}^{t_2} (V_{ab}^b \cdot F_b)\, dt$$

$V_{ac}^b \cdot F_c = V_{ab}^b \cdot F_b$

$\Rightarrow V_{ac}^b \cdot F_c = Ad_{g_{bc}} V_{ac}^b \cdot F_b$ $\quad \{ V_{bc}^b = 0 \}$

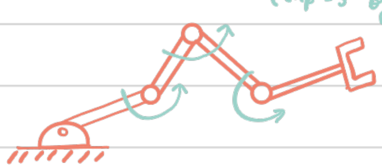
$= (Ad_{g_{bc}} V_{ac}^b)^T F_b$ $\quad V_{ac}^b = Ad_{g_{bc}^{-1}} V_{ab}^b + V_{bc}^b$

$\Rightarrow V_{ac}^b \cdot F_c = V_{ac}^b \cdot Ad_{g_{bc}}^T F_b$

$$\boxed{\begin{bmatrix} f_c \\ \tau_c \end{bmatrix} = \begin{bmatrix} R_{bc}^T & 0 \\ -R_{bc}^T \hat{p}_{bc} & R_{bc}^T \end{bmatrix} \begin{bmatrix} f_b \\ \tau_b \end{bmatrix}}$$

Forward Kinematics

(Chp - 3 of Murray-Li)



Types of joints:

Revolute	θ
Prismatic	θ

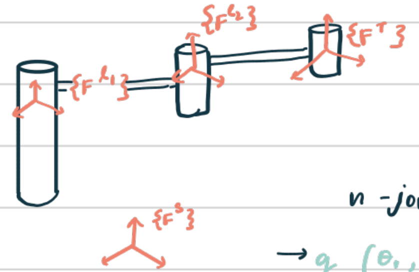
Focus of this course.

Helical

Spherical

Cylindrical

(D-H parameters)



$\pi^2 = S^1 \times S^1$

n-joints

$\rightarrow g_{st}(\theta_1, \cdots \theta_n) = \underbrace{g_{sl_1}}_{\text{Homogeneous}}(\theta_1)\, g_{l_1 l_2}(\theta_2) \cdots g_{l_{n-1} l_n}(\theta_n)\, g_{l_n t}$

Homogeneous matrices.

$g_{ab}(\theta) = e^{\hat{\xi}\theta} g_{ab}(0)$

θ: amount of translation (when a prismatic joint)

θ: amount of rotation (when revolute joint)

Case 1: (Fix 1st Joint, Move 2nd Joint) then (Move 1st Joint)

$g_{st}(\theta_2) = e^{\hat{\xi}_2 \theta_2}$

$g_{st}(\theta_1, \theta_2) = e^{\hat{\xi}_1 \theta_1} g_{st}(\theta_2)$

Case 2: Move 1st then second

$g_{st}(\theta_1) = e^{\hat{\xi}_1 \theta_1} g_{st}(0)$

Now, move 2nd joint: $g_{st}(\theta_1, \theta_2) = e^{\hat{\xi}_2' \theta_2} e^{\hat{\xi}_1 \theta_1} g_{st}(0)$

(Because the twist has changed) $\rightarrow \xi_2' = Ad_{e^{\hat{\xi}_1 \theta_1}} \xi_2$

$\Rightarrow g_{st}(\theta_1, \theta_2) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{-\hat{\xi}_1 \theta} e^{\hat{\xi}_1 \theta} g_{st}(0)$

Both are same.

$\xi_i = \begin{bmatrix} -\omega_i \times q_i \\ \omega_i \end{bmatrix}$ ← Revolute

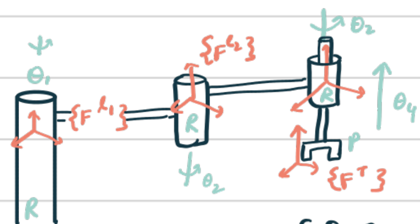
$\xi_i = \begin{bmatrix} v_i \\ 0 \end{bmatrix}$ ← Prismatic

$g : Q \rightarrow SE(3)$

Joint Space

P.O.E. : $g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} \cdots e^{\hat{\xi}_n \theta_n} g_{st}(0)$

RRRP (robot) - Scara



$\omega_1 = \omega_2 = \omega_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$v_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\xi_i = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$ $\quad q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad q_2 = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}$ $\quad q_3 = \begin{bmatrix} 0 \\ l_1 + l_2 \\ 0 \end{bmatrix}$

$\Rightarrow \xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad \xi_2 = \begin{bmatrix} l_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad \xi_3 = \begin{bmatrix} l_1 + l_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad \xi_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} e^{\hat{\xi}_4 \theta_4} g_{st}(0)$

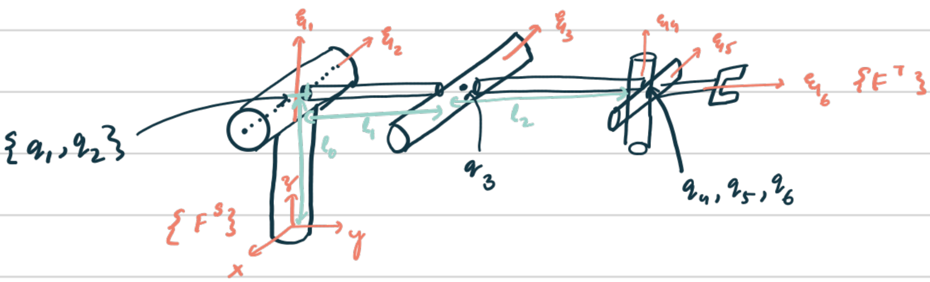
Takes care of the order, we can work with body representation

$$g_{st}(0) = \left[\begin{array}{c|c} I & \begin{matrix} 0 \\ l_1+l_2 \\ l_0 \end{matrix} \\ \hline 0 & 1 \end{array}\right] \; ; \quad e^{\hat{\xi}\theta} = \left[\begin{array}{c|c} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})\,\hat{\omega} v + \omega\omega^T v\theta \\ \hline 0 & 1 \end{array}\right]$$

$$g_{st}(\theta) = \left[\begin{array}{ccc|c} c_{123} & -s_{123} & 0 & -l_1 s_1 - l_2 s_{12} \\ s_{123} & c_{123} & 0 & l_1 c_1 + l_2 c_{12} \\ 0 & 0 & 1 & l_0 + \theta_4 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Elbow Manipulator: Forward Kinematics

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}\, g_{st}(0)$$



$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \omega_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \omega_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \omega_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \omega_5 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \omega_6 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

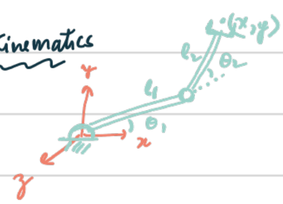
$$g_{st}(0) = \left[\begin{array}{c|c} I & \begin{matrix} 0 \\ l_1+l_2 \\ l_0 \end{matrix} \\ \hline 0 & 1 \end{array}\right]$$

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \xi_2 = \begin{bmatrix} 0 \\ -l_0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \xi_3 = \begin{bmatrix} 0 \\ -l_0 \\ l_1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \xi_4 = \begin{bmatrix} l_1+l_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\xi_5 = \begin{bmatrix} 0 \\ -l_0 \\ l_1+l_2 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \xi_6 = \begin{bmatrix} -l_0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$p(\theta) = \begin{bmatrix} -s_1 (l_1 c_2 + l_2 c_{23}) \\ c_1 (l_1 c_2 + l_2 c_{23}) \\ l_0 - l_1 s_2 - l_2 s_{23} \end{bmatrix}$$

Inverse Kinematics



$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

$$x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 c_2$$

$$\theta_2 = \pm \cos^{-1}(\quad)$$

$\theta_2^1 \rightarrow + \rightarrow$ elbow down solution

$\theta_2^2 \rightarrow - \rightarrow$ elbow up solution

$$\underbrace{\frac{x c_1}{\sqrt{x^2+y^2}}}_{\alpha} + \frac{y}{\sqrt{x^2+y^2}} s_1 = \frac{l_1 + l_2 c_2}{\sqrt{x^2+y^2}}$$

$\tan\alpha$

$$\theta_2^1 \downarrow \theta_1^1 \qquad \theta_2^2 \downarrow \theta_1^2$$

$$\begin{bmatrix} y \\ x \end{bmatrix} = \underbrace{\begin{bmatrix} l_1 + l_2 c_2 & l_2 s_2 \\ -l_2 s_2 & l_1 + l_2 c_2 \end{bmatrix}}_{B} \begin{bmatrix} s_1 \\ c_1 \end{bmatrix}$$

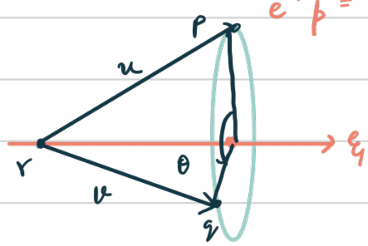
$$B^{-1} = \begin{bmatrix} l_1 + l_2 c_2 & -l_2 s_2 \\ l_2 s_2 & l_1 + l_2 c_2 \end{bmatrix} \frac{1}{l_1^2 + l_2^2 + 2 l_1 l_2 c_2}$$

Paden - Kahan Subproblems

Subproblem 1: Rotation about a single axis.

Let ξ be a 0-pitch twist with unit-magnitude and p & $q \in \mathbb{R}^3$ are two points. Find θ s.t.

$$e^{\hat{\xi}\theta} p = q$$

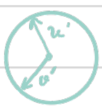


$u = p - r$

$v = q - r$

$e^{\hat{\xi}\theta} r = r$

Projection



$e^{\hat{\omega}\theta} u = v$

$u' = u - \omega\omega^T u$

$v' = v - \omega\omega^T v$

$u' \times v' = \omega \, \|u'\| \, \|v'\| \sin\theta$

$u' \cdot v' = \|u'\| \, \|v'\| \cos\theta$

$$\theta = \text{Atan2}\big(\overset{y}{w^T(u' \times v')},\ \overset{x}{u'^T v'}\big)$$

(points should not be on the axis)

Subproblem 2: Rotation about two subsequent axes.

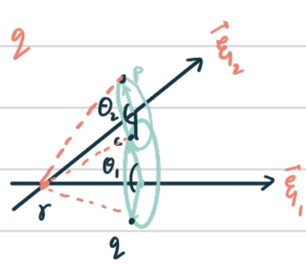
Let ξ_1 & ξ_2 be two zero-pitch unit magnitude twists with intersecting axes and p & $q \in \mathbb{R}^3$. Find θ_1 & θ_2 s.t.

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p = q$$

$u = p - r$

$z = c - r$

$v = q - r$



$$e^{\hat{\omega}_2 \theta_2} u = z = e^{-\hat{\omega}_1 \theta_1} v$$

$$\omega_2^T u = \omega_2^T z \quad ; \quad \omega_1^T v = \omega_1^T z$$

$$\|u\|^2 = \|v\|^2 = \|z\|^2$$

$$z = \alpha \omega_1 + \beta \omega_2 + \gamma (\omega_1 \times \omega_2)$$

$$\omega_1^T z = \alpha + \beta \omega_1^T \omega_2 = \begin{bmatrix} \omega_1^T v \\ \omega_2^T u \end{bmatrix} = \begin{bmatrix} 1 & \omega_1^T \omega_2 \\ \omega_1^T \omega_2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$
$$\omega_2^T z = \alpha \omega_1^T \omega_2 + \beta$$

$$\Rightarrow \begin{bmatrix} \omega_1^T v \\ \omega_2^T u \end{bmatrix} \begin{bmatrix} 1 & -\omega_1^T \omega_2 \\ -\omega_1^T \omega_2 & 1 \end{bmatrix} \cdot \frac{1}{1-(\omega_1^T \omega_2)^2} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\|z\|^2 = \|u\|^2 = \alpha^2 + \beta^2 + 2\alpha\beta\, \omega_1^T \omega_2 + \gamma^2 \|\omega_1 \times \omega_2\|^2$$

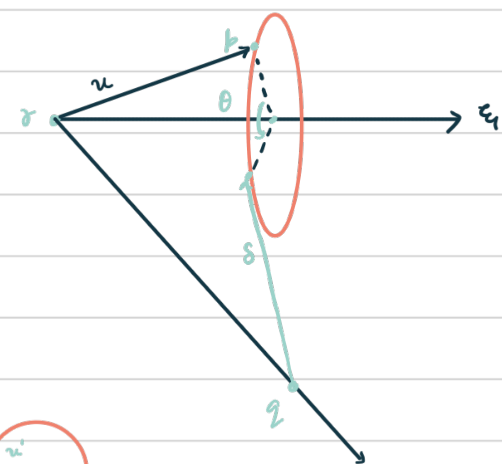
$$\Rightarrow \quad \gamma^2 = \frac{\|u\|^2 - \alpha^2 - \beta^2 - 2\alpha\beta\, \omega_1^T \omega_2}{\|\omega_1 \times \omega_2\|^2}$$

There are two solutions corresponding to $\gamma\{\pm\}$

Subproblem 3: Rotation to a given distance.

Let ξ be a zero pitch, unit magnitude twist with $p, q \in \mathbb{R}^3$ two points. δ be a real number > 0. Find θ s.t.

$$\| q - e^{\hat{\xi}\theta} \| = \delta$$



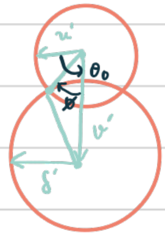
$$u = p - r$$
$$v = q - r$$
$$\| v - e^{\hat{\omega}\theta} u \|^2 = \delta^2$$

$$u' = u - \omega\omega^T u$$

$$v' = v - \omega\omega^T v$$

$$\delta'^2 = \delta^2 - |\omega^T (p - q)|^2$$

$$\| v' - e^{\hat{\omega}\theta} u' \|^2 = \delta'^2$$



$$\theta^1 = \theta_0 + \phi$$
$$\theta^2 = \theta_0 - \phi$$

$$\|u'\|^2 + \|v'\|^2 - 2\|u'\|\|v'\| \cos\phi = \delta'^2$$

$$\phi = \cos^{-1}\left(\frac{\|u'\|^2 + \|v'\|^2 - \delta'^2}{2\|u'\|\|v'\|} \right)$$

$$\theta_0 = \operatorname{Atan2}\left(\omega^T (u' \times v'),\ u'^T v' \right)$$

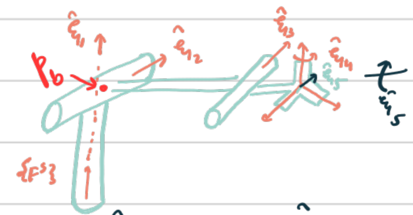
q is in intersection ξ_1 & ξ_2

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p = g_p$$
$$-q \qquad -q$$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} (e^{\hat{\xi}_3\theta_3} p - q) = g_p - q$$

$$\Rightarrow \quad \| e^{\hat{\xi}_3\theta_3} p - q \| = \delta \quad \leftarrow \text{subproblem 3}$$

Elbow Manipulator



$$e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_6\theta_6} \qquad \underbrace{g_{st}(0)}_{\text{known}} = \underbrace{g_{st}(\theta)}_{\text{known}}$$

inv

[First find elbow joint angle - θ_3]

take p_w on intersection of ξ_4, ξ_5, ξ_6

$$\Rightarrow \quad e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} \underbrace{\qquad p_w}_{p_w} = \underbrace{g_{st}(\theta)\, g_{st}^{-1}(0)}_{g}\, p_w$$

$$\Rightarrow \quad e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p_w = g\, p_w$$

$\leftarrow$ good choice of p_b

$$\Rightarrow \quad e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p_w - p_b = g\, p_w - p_b$$

$$\Rightarrow \quad \| e^{\hat{\xi}_3\theta_3} p_w - p_b \| = \| g\, p_w - p_b \|$$

SP 3 $\rightarrow$ θ_3 is found.

$$\theta_3 \Rightarrow \quad e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p = q$$

$(\in e^{\hat{\xi}_3\theta_3} p_w) \qquad (\in q)$

$\hookrightarrow$ SP2 $\longrightarrow$ 2 more solutions.

$\theta_3^1 \quad \theta_3^2$

$\theta_1^1\, \theta_2^1 \quad \theta_1^3\, \theta_2^3$

$\theta_1^2\, \theta_2^2 \quad \theta_1^4\, \theta_2^4$

Use θ_1, θ_2 & θ_3 to simplify the problem

$$e^{\hat{\xi}_4\theta} e^{\hat{\xi}_5\theta} e^{\hat{\xi}_6\theta} = g_1$$

Choose a point p on ξ_6

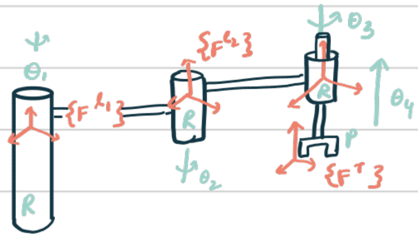
$$e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} \underbrace{e^{\hat{\xi}_6\theta_6} p} = g_1 p$$

Use SP2 to find θ_4 & θ_5 - 2 solutions.

Use SP_1 to find θ_6 ← unique solution.

Total 8 solutions.

SCARA Manipulator



$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} \quad g_{st}(0) = \left[\begin{array}{c|c} R & p \\ \hline 0 & 1 \end{array}\right]$$

$$p = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} \\ l_0 + \theta_4 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \theta_4 \checkmark$$

Goal: Find θ_2 first

Pick p on ξ_3 & q on ξ_1

$$\| \cancel{e^{\hat{\xi}_1\theta_1}}^{I} e^{\hat{\xi}_2\theta_2} \cancel{e^{\hat{\xi}_3\theta_3}}^{I} p - q \| = \| g p - q \|$$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p = g p$$

SP3 to find θ_2

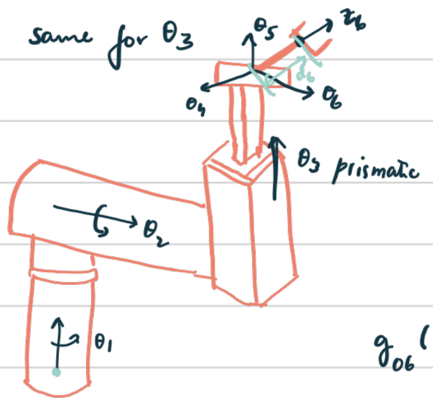
p' on ξ_3 axis

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p' = e^{\hat{\xi}_1\theta_1} \big(\underbrace{e^{\hat{\xi}_2\theta_2} p'}_{q} \big) = g p'$$

SP1 for θ_1 (unique)

Same for θ_3

Stanford arm:



$d_3 \updownarrow \theta_3$

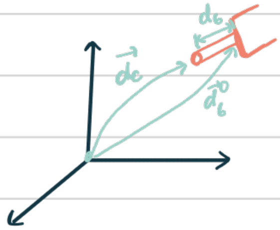
$$g_{06}(\theta) = \left[\begin{array}{c|c} R_6^0 & d_6^0 \\ \hline 0 & 1 \end{array}\right]$$

$$d_6^0 = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$$

$$d_x = d_3 c_1 s_2 - d_2 s_1 + d_6 (c_1 c_2 c_4 s_5 + c_1 c_5 s_2 - s_1 s_4 s_5)$$

$$d_y = d_3 s_1 s_2 + d_2 c_1 + d_6 (c_1 s_4 s_5 + c_2 c_4 s_1 s_5 + c_5 s_1 s_2)$$

$$d_z = d_3 c_2 + d_6 (c_2 c_5 - c_4 s_2 s_5)$$



This decouples the problem allows for a solution.

"Decoupling at wrist"

$$(*)\quad d_6^0 = \underbrace{d_c}_{} + \underbrace{d_6 R_6^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{} = \begin{bmatrix} d_6 r_{13} \\ d_6 r_{23} \\ d_6 r_{33} \end{bmatrix}$$

Can be found out

d_c depends on $\{\theta_1, \theta_2, d_3\}$

$$d_c = \begin{bmatrix} -d_2 s_1 + d_3 c_1 s_2 \\ d_2 c_1 + d_3 s_1 s_2 \\ d_3 c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$d_3^2 s_2^2 = x^2 + y^2$

↳ $x^2+y^2+z^2 = d_2^2 + d_3^2$ → gives d_3

↳ $d_3 c_2 = z$ → c_2 gives 2 solutions for θ_2

↳ Using θ_2 : $\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} -d_2 & d_3^{var} s_2 \\ d_3^{var} s_2 & d_2 \end{bmatrix}}_{} \begin{bmatrix} s_1 \\ c_1 \end{bmatrix}$

det $= -(d_2^2 + d_3^2 s_2^2) \neq 0$

d_3

θ_2^1 → θ_1^1 → θ_5^1, θ_5^2; θ_4^1; θ_6^1

θ_2^2 → θ_1^2 → θ_5^3, θ_5^4; θ_4^2; θ_6^2

$$\underbrace{R_6^0}_{\text{Known}} = \underbrace{R_3^0}_{\text{fixed}} \underbrace{R_6^3}_{\text{to find}}$$

$$R_6^3 = (R_3^0)^{-1} R_6^0$$

$$= \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 c_6 - s_4 c_6 & c_4 s_5 \\ s_4 c_5 c_6 + c_4 s_6 & -c_5 s_6 s_4 + c_4 c_6 & s_4 s_5 \\ -s_5 c_6 & s_5 s_6 & c_5 \end{bmatrix}$$

using these 3 find out $\theta_4, \theta_5, \theta_6$

(There will be a multiplicity introduced. Similar to the Euler angle approach)

$\theta_5^1 = \cos^{-1}(r_{33}')$

$\theta_5^2 = -\cos^{-1}(r_{33}')$

$\theta_4 = \text{Atan2}(r_{23}', r_{13}')$

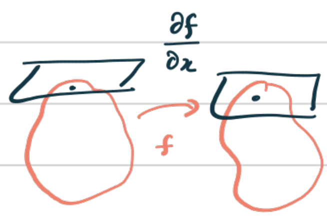
$\theta_6 = \text{Atan2}(r_{32}', -r_{31}')$

If $\theta_5 = 0$ or π, we get $\theta_4 + \theta_6 = \text{const}$ or $\theta_4 - \theta_6 = \text{const}$ } → Infinite Solutions.

Jacobian:

$f: \underbrace{G}_{\mathbb{R}^n} \to \underbrace{H}_{SE(3)}$

$J: \dfrac{\partial f}{\partial x_i} : \dot{\theta} \to V$



$V = J\dot{\theta}$

$$\hat{V}^s_{st} = \dot{g}_{st}(\theta)\, g^{-1}_{st}(\theta)$$

$$= \sum_{i=1}^{n} \left(\frac{\partial g_{st}}{\partial \theta_i} \dot{\theta}_i \right) g^{-1}_{st}(\theta) \qquad = [\xi_1 \; \xi_2' \; \cdots \; \xi_n'] \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \cdots e^{\hat{\xi}_i\theta_i} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0)$$

$$\frac{\partial g_{st}}{\partial \theta_i}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \cdots e^{\hat{\xi}_{i-1}\theta_{i-1}} \hat{\xi}_i e^{\hat{\xi}_i\theta_i} e^{\hat{\xi}_{i+1}\theta_{i+1}} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0)$$

$$\frac{\partial g_{st}}{\partial \theta_i}(\theta)\, g^{-1}_{st}(\theta) = e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_{i-1}\theta_{i-1}} \hat{\xi}_i e^{-\hat{\xi}_{i-1}\theta_{i-1}} \cdots e^{-\hat{\xi}_1\theta_1}$$

$$= \left(Ad_{e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_{i-1}\theta_{i-1}}} \xi_i \right)^\wedge$$

$$J^s_{st}(\theta) = [\xi_1 \; \xi_2' \; \cdots \; \xi_n']$$

$$\underset{\sim}{\tau} \quad \boxed{[-\omega' \times q', \; \omega']}$$

↑ spatial Manipulator problem

Body Manipulator Jacobian

$$J^b_{st}$$

$$\hat{V}^b_{st} = g^{-1} \dot{g}$$

$$g^{-1} \frac{\partial g}{\partial \theta_i} \dot{\theta}_i = g^{-1}_{st}(0)\, e^{-\hat{\xi}_n\theta_n} e^{-\hat{\xi}_{n-1}\theta_{n-1}} \cdots e^{-\hat{\xi}_1\theta_1} \cdot \left(e^{\hat{\xi}_1\theta_1} \cdots \hat{\xi}_i e^{\hat{\xi}_i\theta_i} e^{\hat{\xi}_{i+1}\theta_{i+1}} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0) \right)$$

$$= Ad_{g^{-1}_{st}(0)\, e^{-\hat{\xi}_n\theta_n} \cdots e^{-\hat{\xi}_i\theta_i}} \hat{\xi}_i$$

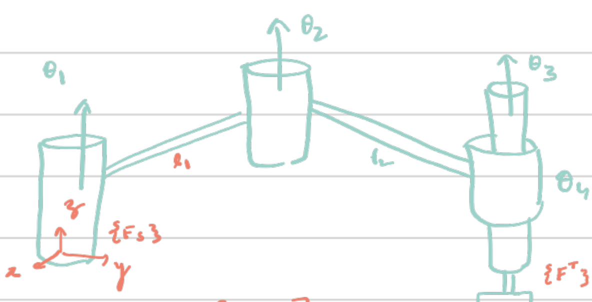
$$\boxed{\xi_i^\dagger = Ad^{-1}_{e^{\hat{\xi}_i\theta_i} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0)} \hat{\xi}_i}$$

$$J^b_{st}(\theta) = [\xi_1^\dagger \; \cdots \; \xi_n^\dagger]$$

$$\boxed{\left(V^s_{st} = Ad_{g_{st}(\theta)} V^b_{st} \right)}$$

Examples:

SCARA



$w_i \leftarrow$ remains unchanged

$q_i \leftarrow$ changes.

Find q_i dependent on θ_i, take cross product.

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \xi_2' = \begin{bmatrix} l_1\cos\theta_1 \\ l_1\sin\theta_1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$-\omega \times q: \quad \begin{matrix} -l_1\sin\theta_1 \\ l_1\cos\theta_1 \\ 0 \end{matrix} \quad \begin{matrix} 0 \\ 0 \\ 1 \end{matrix}$$

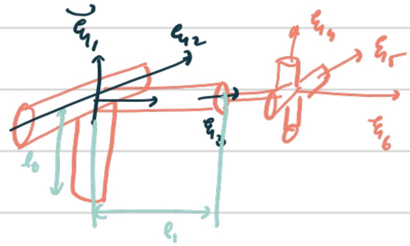
$$\xi_3' = \begin{bmatrix} l_1c_1 + l_2c_{12} \\ l_1s_1 + l_2s_{12} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \xi_4' = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$-l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2)$

$l_1 c_1 + l_2 c_{12}$

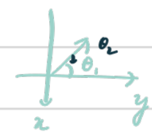
$$J_{st}^s = \begin{bmatrix} 0 & l_1c_1 & l_1c_1+l_2c_{12} & 0 \\ 0 & l_1s_1 & l_1s_1+l_2s_{12} & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Stanford Arm



$$q_2 = \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \qquad \omega_2 = \begin{bmatrix} -\cos\theta_1 \\ -\sin\theta_1 \\ 0 \end{bmatrix}$$

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \xi_2' = \begin{bmatrix} l_0s_1 \\ -l_0c_1 \\ 0 \\ -c_1 \\ -s_1 \\ 0 \end{bmatrix}$$



$$\xi_3' = \begin{bmatrix} -\sin\theta_1 \cos\theta_2 \\ \cos\theta_1 \cos\theta_2 \\ -\sin\theta_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{pmatrix} -s_1c_2 \\ c_1c_2 \\ -s_2 \end{pmatrix} \qquad \begin{matrix} 0 \\ 1 \\ 0 \end{matrix}$$

wrist location

$\xi_4' =$

$q_w' = (-\omega_4' \times q_w', \ \omega_4')$

$$q_w' = \begin{bmatrix} 0 \\ 0 \\ l_w \end{bmatrix} + e^{\hat{z}\theta_1} e^{-\hat{x}\theta_2} \begin{bmatrix} 0 \\ l_1+\theta_3 \\ 0 \end{bmatrix}$$

$$\omega_4' = e^{\hat{z}\theta_1} e^{-\hat{x}\theta_2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -s_1s_2 \\ c_1s_2 \\ c_2 \end{bmatrix} \qquad q_w' = \begin{bmatrix} -(l_1+\theta_3)\, s_1c_2 \\ (l_1+\theta_3)\, c_1c_2 \\ l_0 - (l_1+\theta_3)\, s_2 \end{bmatrix}$$

$$\omega_5' = e^{\hat{z}\theta_1} e^{-\hat{x}\theta_2} e^{\hat{z}\theta_4} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_1c_4 + s_1c_2s_4 \\ -s_1c_4 - c_0c_2s_4 \\ s_2s_4 \end{bmatrix}$$

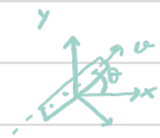
$$\omega_6' = e^{\hat{z}\theta_1} e^{-\hat{x}\theta_2} e^{\hat{z}\theta_4} e^{-\hat{x}\theta_5} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_5(s_1c_2c_4 + c_1s_4) + s_1s_2s_5 \\ c_5(c_1c_2c_4 - s_1s_4) - c_1s_2s_5 \\ -s_2c_4c_5 - c_2s_5 \end{bmatrix}$$

$$J_{st}^s = \begin{bmatrix} 0 & -\omega_2' \times q_2' & v_3' & -\omega_4' \times q_4' & -\omega_5' \times q_5' & -\omega_6' \times q_6' \\ \omega_1 & \omega_2' & 0 & \omega_4' & \omega_5' & \omega_6' \end{bmatrix}$$

Dynamics

Generalized coordinates $\leftarrow$ Minimum no. of coordinates needed to define the phase space.

Non-holonomic constraint :



$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$$

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$

(cannot be integrated to the form $\otimes$ $g(x,y,\theta) = 0$

Lagrange's Equations of motion

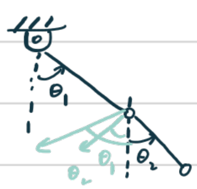
$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \gamma_i^s} \quad \{i = 1, \dots, n\}$$

$q_i \leftarrow$ Generalized coordinates

$\gamma_i^s \leftarrow$ Generalized forces (corresponding to gen. coordinate)

$$\mathcal{L} = \underset{\substack{\uparrow \\ KE}}{T} - \underset{\substack{\uparrow \\ PE}}{V}$$

Analytical Dynamics of Discrete Systems $\leftarrow$ Great Book



$$T = -m_1 g_1 l_1 \cos\theta_1 - m_2 g_2 (l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$V = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right)$$

Spherical Pendulum

$$\mathcal{L} = T - V$$

$$T = \frac{1}{2} m v^2 \qquad V = mgh$$

$$x = l\sin\theta\cos\phi \qquad \dot{x} = l\cos\theta\,\dot{\theta}\cos\phi - l\sin\theta\sin\phi\,\dot{\phi}$$

$$y = l\sin\theta\sin\phi \qquad \dot{y} = l\cos\theta\,\dot{\theta}\sin\phi + l\sin\theta\cos\phi\,\dot{\phi}$$

$$z = -l\cos\theta \qquad \dot{z} = l\sin\theta\,\dot{\theta}$$

$$v^2 = l^2\dot{\theta}^2 + l^2\dot{\phi}^2\sin^2\theta$$

$$T = -mgl\cos\theta$$

$$\mathcal{L} = T - V = \frac{1}{2} m l^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + mgl\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot\theta} = m l^2 \dot\theta \qquad \frac{\partial \mathcal{L}}{\partial \dot\phi} = m l^2 \sin^2\theta \, \dot\phi$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = m l^2 \dot\phi^2 \sin\theta \cos\theta - mgl \sin\theta \qquad \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\theta \rightarrow \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \quad \rightarrow \quad m l^2 \ddot\theta = m l^2 \dot\phi^2 s_\theta c_\theta - mgl s_\theta$$

$$\phi \rightarrow \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \rightarrow \quad 2 \sin\theta \cos\theta \, \dot\theta \dot\phi + \sin^2\theta \, \ddot\phi = 0$$

$$\underbrace{\begin{bmatrix} ml^2 & 0 \\ 0 & ml^2 \sin^2\theta \end{bmatrix}}_{\text{Mass matrix is symmetric}} \begin{bmatrix} \ddot\theta \\ \ddot\phi \end{bmatrix} + \begin{bmatrix} -m l^2 \dot\phi^2 \sin\theta\cos\theta \\ 2 m l^2 \dot\phi \dot\theta \sin\theta \cos\theta \end{bmatrix} + \begin{bmatrix} mgl \sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Try for simple cases like $\dot\phi = 0$

(Gravitional term - coefficient +ve) ?

Find D.E. of motion of a particle in a uniform gravitional field.

Spherical Coordinates : r, θ, ϕ

$$T = \frac{1}{2} m \left(\dot r^2 + r^2 \dot\theta^2 + r^2 \sin^2\theta \, \dot\phi^2\right)$$

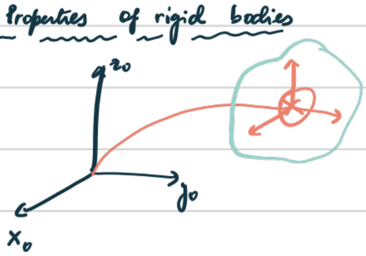
$$V = mgr \cos\theta$$

r : $\ddot r - r\dot\theta^2 - r\dot\phi^2 \sin^2\theta + g\cos\theta = 0$

θ : $r\ddot\theta + 2\dot r \dot\theta - r\dot\phi^2 \sin\theta\cos\theta - g\sin\theta = 0$

ϕ : $r\ddot\phi \sin\theta + 2\dot r \dot\phi \sin\theta + 2r\dot\theta\dot\phi\cos\theta = 0$

Inertial Properties of rigid bodies



$$p = Rr + p$$

$$\dot p = \dot R r + \dot p$$

$$m = \int_V \rho(v)\, dv$$

$$\bar r = \frac{1}{m} \int_V r \rho(v)\, dv$$

$$T : \frac{1}{2} m v^2 \rightarrow \int_V \frac{1}{2} \rho(r) \underbrace{\|\dot R r + \dot p\|^2}_{= 2\dot p^T \dot R r + \dot p^T \dot p + r^T \dot R^T \dot R r} dv$$

$dv = dx\,dy\,dz$

$$T = \frac{1}{2}\int_V \rho(r)\left(\|\dot p\|^2 + \underbrace{2\dot p^T \dot R r}_{\downarrow} + \|\dot R r\|^2\right) dv$$

$$\left(\dot p^T \dot R \int \underbrace{\rho(r)\, r\, dv}_{\text{com is defined appropiately.}} = 0\right)$$

$$\frac{1}{2}\int \rho(r)\; r^T \dot R^T \dot R\, r\, dv \qquad (r\times\omega)^T (r\times\omega)$$

$$\dot R = R\hat\omega^b \quad = \int \rho(r)\; \underbrace{r^T \hat\omega^{bT}\hat\omega^b r}_{\downarrow}\, dv$$

$$\omega^{bT}\, \hat r\, \hat r^T \omega^b\, dv$$

$$= \frac{1}{2}\int \rho\; (\hat r\omega)^T(\hat r\omega)\, dr$$

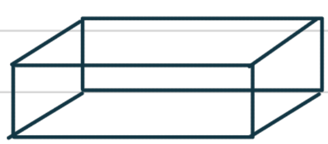
$$= \frac{1}{2}\,\omega^T \underbrace{\left[\int \rho(r)\, \hat r^T \hat r\, dv\right]}_{\mathcal{I}} \omega \quad := \frac{1}{2}\,\omega^T \mathcal{I}\,\omega$$

$$\hat r = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \qquad \hat r^2 = \begin{bmatrix} -y^2-z^2 & xy & xz \\ \sim & -x^2-z^2 & \\ & & -x^2-y^2 \end{bmatrix}$$

$$\boxed{\begin{aligned} I_{xx} &= \int \rho(r)\,(y^2+z^2)\,dv \\ I_{xy} &= -\int \rho(v)\,(xy)\,dv \end{aligned}}$$

$$\hat V^b = g^{-1}\dot g$$

$$T = \frac{1}{2}(V^b)^T \begin{bmatrix} mI & 0 \\ 0 & \mathcal{I} \end{bmatrix} V^b := \frac{1}{2}(V^b)^T \underbrace{\mathcal{M}}_{\text{Generalized Interia matrix}} V^b \qquad \left[\begin{array}{c|c} \begin{smallmatrix} m & & 0 \\ & m & \\ 0 & & m \end{smallmatrix} & 0 \\ \hline 0 & \begin{smallmatrix} I_{xx} & & \\ & I_{yy} & \\ & & \ddots \end{smallmatrix} \end{array}\right]$$



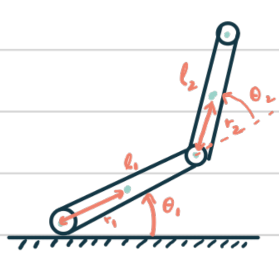
$$I_{xx} = \int_{-l/2}^{l/2}\int_{-w/2}^{w/2}\int_{-h/2}^{h/2} (y^2+z^2)\,dx\,dy\,dz \cdot \frac{m}{lwh}$$

$$= \frac{m}{lwh}\cdot l \cdot 2\left(\frac{w^3}{3\times 8}h + \frac{h^3}{3\times 8}w\right)$$

$$= \frac{m}{12}(h^2+w^2)$$

$$I_{xy} = 0$$

$$\mathcal{M} = \left[\begin{array}{c|c} \begin{smallmatrix} m & & 0 \\ & m & \\ 0 & & m \end{smallmatrix} & 0 \\ \hline 0 & \mathcal{I} \end{array}\right]$$



$$I = \begin{bmatrix} I_{x_i} & & 0 \\ & I_{y_i} & \\ 0 & & I_{z_i} \end{bmatrix}$$

$$T(\theta,\dot\theta) = ?$$

$$= \frac{1}{2} I_{z_1}\,\omega_1^2 + \frac{1}{2} m_1 v_1^2$$

$$+ \frac{1}{2} I_{z_2}(\omega_1+\omega_2)^2 + \frac{1}{2} m_2 v_2^2$$

$$v_1 = r_1\dot\theta_1, \quad v_2^2 = l_1^2\dot\theta_1^2 + r_2^2(\dot\theta_1+\dot\theta_2)^2 + 2 l_1 r_2 \cos\theta_2\, \dot\theta_1(\dot\theta_1+\dot\theta_2)$$

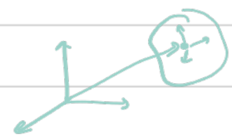
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta_1}\right) - \frac{\partial T}{\partial \theta_1} = \tau_1 \quad ; \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta_2}\right) - \frac{\partial T}{\partial \theta_2} = \tau_2$$

$$\begin{bmatrix} \alpha + 2\beta c_2 & \delta + \beta c_2 \\ \delta + \beta c_2 & \delta \end{bmatrix} \begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{bmatrix} + \begin{bmatrix} -\beta s_2 \dot\theta_2 & -\beta s_2 (\dot\theta_1 + \dot\theta_2) \\ \beta s_2 \dot\theta_1 & 0 \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$$\alpha = I_{z_1} + I_{z_2} + m_1 r_1^2 + m_2(l_1^2 + r_2^2)$$
$$\beta = m_2 l_1 r_2$$
$$\delta = I_{z_2} + m_2 r_2^2$$

Newton-Euler Equations of motion:

$$f = \frac{d}{dt}(m\dot p)$$
$$\tau = \frac{d}{dt}(I'\omega^s)$$



$g : (p, R) \longrightarrow SE(3)$

$$I' = R I R^T \qquad \frac{dI}{dt} = 0$$

$$\frac{d}{dt}(I'\omega^s) = \hat\omega^s I'\omega^s + I\hat\omega^s\omega^s + I'\dot\omega^s$$

$$\tau = I'\dot\omega^s + \omega^s \times I'\omega^s$$

Body frame:

$$v^b = R^T \dot p$$
$$\omega^b = R^T \omega^s$$
$$f^b, \tau^b$$

$$\begin{bmatrix} mI & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot v^b \\ \dot\omega^b \end{bmatrix} + \begin{bmatrix} \omega^b \times m v^b \\ \omega^b \times I\omega^b \end{bmatrix} = \begin{bmatrix} R^T f \\ R^T \tau \end{bmatrix} = \begin{bmatrix} f^b \\ \tau^b \end{bmatrix} = F^b$$

$$\frac{d}{dt}(m\dot p) = \frac{d}{dt}(m R v^b) \qquad f^b = m\dot v^b + \omega^b \times m v^b$$

Dynamics of open-chain manipulators in joints

L_i : Coordinate frame for each link at the COM of the link

g_{sL_i} : configuration of frame L_i wrt S.

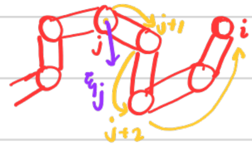
$$g_{sL_i} = e^{\hat\xi_1\theta_1} e^{\hat\xi_2\theta_2} \cdots e^{\hat\xi_i\theta_i} g(0)$$

Body velocity of COM of the i^{th} link $\quad V^b_{sL_i} = J^b_{sL_i}(\theta)\cdot\dot\theta$

(sub summed)

$$J^b_{sL_i}(\theta) = \left[\xi_1^{\dagger i} \; \xi_2^{\dagger i} \cdots \xi_i^{\dagger i} \; 0 \; 0 \cdots 0\right]$$

$$\xi_j^{\dagger i} = Ad^{-1}\left(e^{\hat{\xi}_j\theta_j} \cdots e^{\hat{\xi}_i\theta_i}\, g_{sl_i}(0)\right) \underbrace{\xi_j}_{\text{from } j \text{ to } i} \quad j \quad j \le i$$



$\xi_j^\dagger$ is the instantaneous twist relative to the i^{th} link frame

$J^b_{sl_i} = J_i$ for notation

$$T_i(\theta, \dot\theta) = \frac{1}{2}\left(V^b_{sl_i}\right)^T \mu_i \left(V^b_{sl_i}\right) = \frac{1}{2}\dot\theta^T J_i^T \mu_i J_i \dot\theta$$

↳ $= J_i \dot\theta$ depends on the velocity of all previous joints.

$$T(\theta, \dot\theta) = \sum_{i=1}^{n} T_i(\theta, \dot\theta) = \frac{1}{2}\dot\theta^T M(\theta)\, \dot\theta$$

$$M(\theta) = \sum_{i=1}^{n} J_i^T(\theta)\, \mu_i\, J_i(\theta)$$

↳ $\in \mathbb{R}^{n\times n}$ manipulator inertia matrix

$V_i(\theta) \Rightarrow$ Potential Energy

$$V = \sum_{i=1}^{n} V_i(\theta) = \sum_{i=1}^{n} m_i\, g\, h_i(\theta)$$

$$\mathcal{L}(\theta, \dot\theta) = T(\theta, \dot\theta) - V(\theta) = \frac{1}{2}\dot\theta^T M(\theta)\, \dot\theta - V(\theta)$$

Equations of motion: $\underbrace{\frac{d}{dt}\left(\frac{\partial}{\partial\dot\theta}\mathcal{L}\right)} - \underbrace{\frac{\partial}{\partial\theta}\mathcal{L}} = \tau_i$

$$\frac{d}{dt}\left(\sum M_{ij}\dot\theta_j\right) = \left(\sum_{j=1}^{n}\left(\dot M_{ij}\dot\theta_j + M_{ij}\ddot\theta_j\right)\right)$$

$$\frac{1}{2}\dot\theta^T\left(\frac{\partial}{\partial\theta_i}M(\theta)\right)\dot\theta - \frac{\partial}{\partial\theta_i}(V(\theta))$$

$$\Rightarrow \sum_{j=1}^{n} M_{ij}\ddot\theta_j + \underbrace{\sum_{j,k=1}^{n}}\left(\frac{\partial M_{ij}}{\partial\theta_k}\dot\theta_k\dot\theta_j - \frac{1}{2}\frac{\partial M_{kj}}{\partial\theta_i}\dot\theta_k\dot\theta_j\right) + \frac{\partial V}{\partial\theta_i} = \tau_i \quad \{i = 1, \dots, n\}$$

$$\hookrightarrow \frac{1}{2}\sum \left(M_{ij,k} + M_{ik,j} - M_{kj,i}\right)\dot\theta_j\dot\theta_k$$

Γ_{ijk} Christoff's Symbols

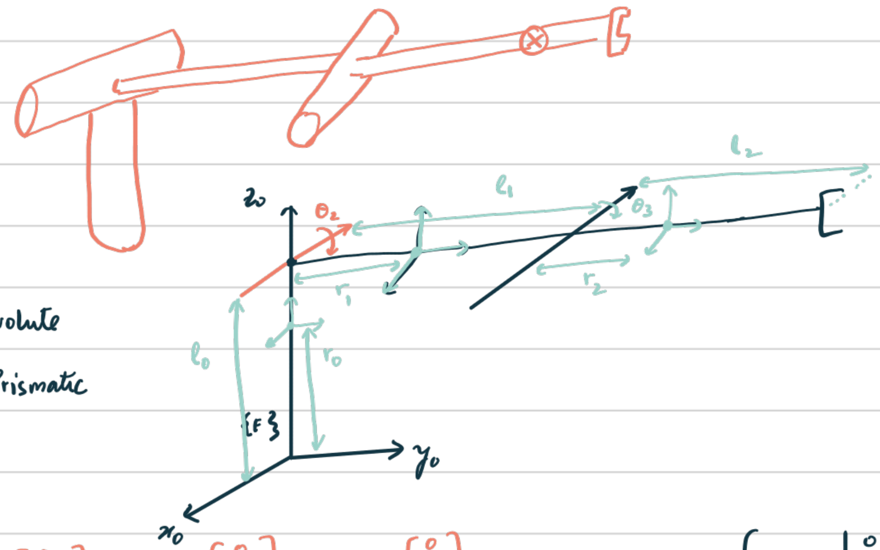
$$-N(\theta, \dot\theta) = \frac{-\partial V}{\partial\theta_i} - \beta\,\dot\theta_i$$

damping

$$M(\theta, \dot\theta)\,\ddot\theta + C(\theta, \dot\theta)\,\dot\theta + N(\theta, \dot\theta) = \tau$$

$$C(\theta, \dot\theta) = \sum_{i=1}^{n}\Gamma_{ijk}\dot\theta_k = \frac{1}{2}\sum_{k=1}^{n}\left(M_{ij,k} + M_{ik,j} - M_{kj,i}\right)\dot\theta_k$$

3- link open chain manipulator



⊙ : Revolute

Prismatic

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 0 \\ -l_0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \xi_3 = \begin{bmatrix} 0 \\ -l_0 \\ l_1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$g_{sl_1}(0) = \left[\begin{array}{c|c} I & \begin{matrix} 0 \\ 0 \\ r_0 \end{matrix} \\ \hline 0 & 1 \end{array}\right]$$

$$g_{sl_2}(0) = \left[\begin{array}{c|c} I & \begin{matrix} 0 \\ r_1 \\ l_0 \end{matrix} \\ \hline 0 & 1 \end{array}\right]$$

$$g_{sl_3}(0) = \left[\begin{array}{c|c} I & \begin{matrix} 0 \\ l_1 + r_2 \\ l_0 \end{matrix} \\ \hline 0 & 1 \end{array}\right]$$

$$M_i = \left[\begin{array}{ccc|ccc} m_i & 0 & 0 & & & \\ 0 & m_i & 0 & & 0 & \\ 0 & 0 & m_i & & & \\ \hline & & & I_{x_i} & 0 & 0 \\ & 0 & & 0 & I_{y_i} & 0 \\ & & & 0 & 0 & I_{z_i} \end{array}\right]$$

$$J_1 = J^b_{sl_1}(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad J_2 = J^b_{sl_2}(0) = \begin{bmatrix} -r_1 l_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -r_1 & 0 \\ 0 & -1 & 0 \\ -s_2 & 0 & 0 \\ c_2 & 0 & 0 \end{bmatrix}$$

$$J_3 = J^b_{sl_3}(0) = \begin{bmatrix} -l_2 c_2 - r_2 c_{23} & 0 & 0 \\ 0 & l_1 s_3 & 0 \\ 0 & -r_2 - l_1 c_3 & -r_2 \\ 0 & -1 & -1 \\ -s_{23} & 0 & 0 \\ c_{23} & 0 & 0 \end{bmatrix}$$

μ/M : $$M(\theta) = J_1^T \mu_1 J_1 + J_2^T \mu_2 J_2 + J_3^T \mu_3 J_3$$

$$M(\theta) = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$M_{11} = I_{y_2} s_2^2 + I_{y_3} s_{23}^2 + I_{z_1} + I_{z_2} c_2^2 + I_{z_3} c_{23}^2 + m_2 r_1^2 c_2^2 + m_3 (l_1 c_2 + r_2 c_{23})^2$$

$$M_{12} = M_{13} = 0 = M_{21} = M_{31}$$

$$M_{22} = I_{x_2} + I_{x_3} + m_3 l_1^2 + m_2 r_1^2 + m_3 r_2^2 + m_3 r_2^2 + 2m_3 l_1 r_2 c_3$$

$$M_{23} = I_{x_3} + m_3 r_2^2 + m_3 l_1 r_2 c_3$$

$$M_{33} = I_{x_3} + m_2 r_2^2$$

Does not have terms from prior links.

3 link: 27 gammas

$$\Gamma_{ijk} = \frac{1}{2} \left(M_{ij,k} + M_{ik,j} - M_{kj,i} \right)$$

$$\Gamma_{131} = \frac{1}{2} (M_{11,3}) = \dots$$

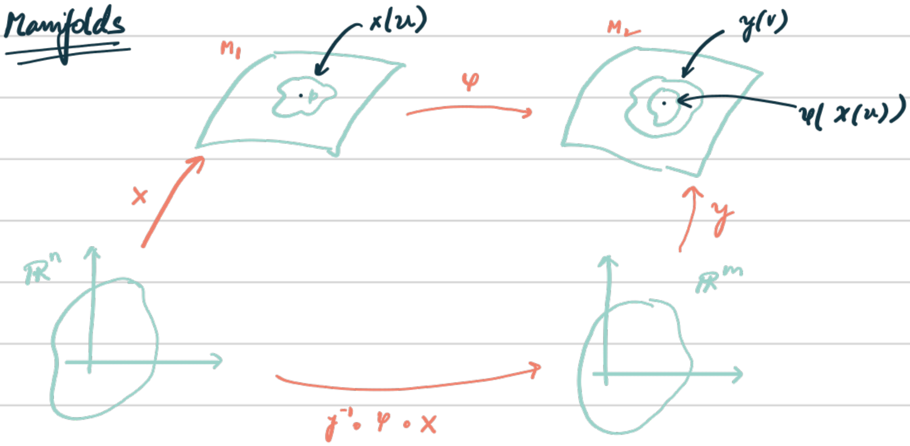
Non zeros: 112, 113, 121, 131, 211, 223, 232, 233, 311, 322

$$N = \frac{\partial V}{\partial \theta}, \quad h_i(\theta) \longrightarrow \text{Configuration Dependent}$$

$$h_1(\theta) = r_0 \; ; \; h_2(\theta) = l_0 - r_1 s_2 \; ; \; h_3(\theta) = l_0 - l_1 s_2 - r_2 s_{23}$$

Forward Kinematics

Manifolds



Definition: Let M_1^n & M_2^m be differentiable manifolds

A mapping $\varphi: M_1 \longrightarrow M_2$ is differentiable at $p \in M_1$

if a given parametrization $y: V \subset \mathbb{R}^m \longmapsto M_2$ at $\varphi(p)$

there exists a parametrization $x: U \subset \mathbb{R}^n \longmapsto M_1$ at p

s.t. $\varphi(x(u)) \subset y(V)$ and the mapping

$$y^{-1}\circ\varphi\circ x : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \text{is diff. at } x^{-1}(p).$$

Definition: φ is diff. on an open set of M_1 if it is differentiable at all points of this open set.

The mapping $y^{-1}\circ\varphi\circ x$ is called an expression of φ in the parametrization x & y.

Tangent vectors: In $\mathbb{R}^3$, tangent vector to this curve, v, that is at some point p on the surface.

Let $\alpha: (-\varepsilon, \varepsilon) \longmapsto \mathbb{R}^n$ be a differentiable curve in $\mathbb{R}^n$ with $\alpha(0) = p$

$$\alpha(t) = (x_1(t), x_2(t), \cdots, x_n(t))$$

$$t \in (-\varepsilon, \varepsilon) \quad ; \quad (x_1, x_2, \cdots, x_n) \subset \mathbb{R}^n$$

$$\alpha'(0) = (\dot{x}_1(0), \cdots, \dot{x}_n(0)) = v \in \mathbb{R}^n$$

Let f be a differentiable function defined in a neighbourhood of p.

we can restrict f to the curve α & express the directional derivative wrt vector $v \in \mathbb{R}^n$ as:

$$\frac{d}{dt}(f\circ\alpha)\Big|_{t=0} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Big|_{t=0} \frac{dx_i}{dt}\Big|_{t=0} = \left(\sum x_i'(0)\frac{\partial}{\partial x_i}\right) f$$

So the directional derivative wrt v is an operator on differentiable functions that depend uniquely on v.

Real Analysis by Rudin

Complex Analysis - Alford

Definition: Let M be a differentiable manifold. A differentiable function $\alpha : (-\varepsilon, \varepsilon) \to M$ is called a differentiable curve in M.

Suppose $\alpha(0) = p \in M$. Let $\mathcal{D}$ be the set of all functions on M that are differentiable at p.

The tangent vector to the curve α at $t=0$ is a function $\alpha'(0) : \mathcal{D} \to \mathbb{R}$

Given by
$$\alpha'(0) f = \frac{d}{dt}(f \circ \alpha)\Big|_{t=0} \qquad f \in \mathcal{D}$$

A tangent vector at p is the tangent vector at $t=0$ for some curve $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$. The set of all such tangent vectors to M at p will be indicated by $T_p M$.

If we can choose a parametrization $\mathbf{x} : U \to M^*$ at $p = \mathbf{x}(0)$ we can express the function f & the curve $\mathcal{L}$ in this parametrization by:

$$f \circ \mathbf{x}(q) = f(x_1, \cdots, x_n) \qquad q = (x_1, \cdots, x_n) \in U$$

$$\text{and} \quad \mathbf{x}^{-1} \circ \alpha(t) = (x_1(t), \dots, x_n(t)) \quad \text{respectively}$$

Restricting f to α, we obtain:

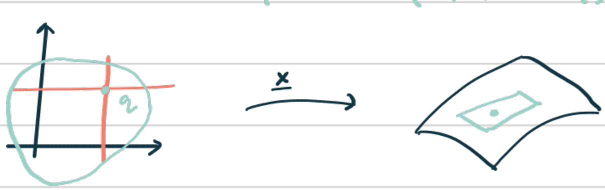
$$\alpha'(0) f = \frac{d}{dt}(f \circ \alpha)\Big|_{t=0} = \frac{d}{dt}\Big(f(x_1(t), \cdots, x_n(t))\Big)\Big|_{t=0}$$

$$= \sum_{i=1}^{n} x_i'(0) \frac{\partial f}{\partial x_i} = \left(\sum_{i=1}^{n} x_i'(0) \left(\frac{\partial}{\partial x_i}\right)_0\right) f$$

The vector $\alpha'(0)$ can be expressed in the parametrization x by $\alpha'(0) = \sum_{i=1}^{n} x_i'(0) \left(\frac{\partial}{\partial x_i}\right)$

$\left(\frac{\partial}{\partial x_i}\right)_0$ is the tangent vector at p of the coordinate curve:

$$x_i \longmapsto \mathbf{x}(0, 0, \cdots 0, x_i, 0 \cdots, 0)$$



$$\left\{ \left(\frac{\partial}{\partial x_1}\right)_0, \cdots \left(\frac{\partial}{\partial x_n}\right)_0 \right\} \in T_p M$$

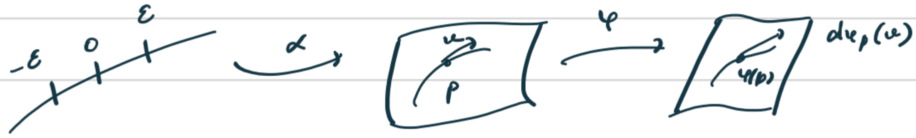
Linear structure in $T_p M$

Let M_1^n & M_2^n be differentiable manifolds and $\varphi: M_1 \to M_2$ be a differentiable mapping

For every $p \in M_1$ & for each $v \in T_p M_1$, choose a differentiable curve $\alpha: (-\varepsilon, \varepsilon) \to M_1$ with $\alpha(0) = p$, $\alpha'(0) = v$

Take $\vec{\beta} = \varphi \cdot \alpha$ The mapping $d\varphi_p: T_p M_1 \longrightarrow T_{\varphi(p)} M_2$

Given by $d\varphi_p(v) = \beta'(0)$ is a linear mapping that does not depend on the choice of α.



$d\varphi_p$ is called the differential of φ at p.

Vector fields & Brakets

Definition: A vector field X on a differentiable manifold M is a correspondance that associates to each point $p \in M$, a vector $X(p) \in T_p M$.

X is a mapping of M into the tangent bundle TM.

The v.f. is differentiable if the mapping $X: M \to TM$ is differentiable.

$$TM = \{(p, v) \,;\, p \in M,\ v \in T_p M\}$$

Consider some parametrization $\underline{x}: U \subset \mathbb{R}^n \longrightarrow M$ we can write

$X(p) = \sum_{i=1}^{n} a_i(p) \dfrac{\partial}{\partial x_i}$ where each $a_i: U \to \mathbb{R}$ is a function on U & $\left\{\dfrac{\partial}{\partial x_i}\right\}$ is a basis associated with $\underline{x}$.

X is differentiable iff a_i's are differentiable.

$X: \mathcal{D} \longmapsto F$

Diff. functions on n

$$(X f)(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial f}{\partial x_i}(p)$$

If $\varphi: M \to N$ is a diffeomorphism, $v \in T_p M$ and f is differentiable function in a nbdh of $\varphi(p)$

then $(d\varphi(v)\, f)_{\varphi(p)} = v\,(f \circ \varphi)(p)$

Let $\alpha : (-\varepsilon, \varepsilon) \to M$ be differentiable curve with $\alpha'(0) = v$, $\alpha(0) = p$, then

$$(d\varphi(v) f)_{\varphi(p)} = \frac{d}{dt}(f \circ \varphi \circ \alpha)\Big|_{t=0} = v(f \circ \varphi)_p$$

Lemma: Let X & Y be differentiable v.f. on a differentiable manifold M. Then there exist a unique v.f. Z s.t. $\forall f \in \mathcal{D}$

$$Zf = (XY - YX) f$$

Let $p \in M$ & let $x : U \to M$ be a parametrization at p.

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \qquad Y = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}$$

$$XY f = X(Yf) = X \sum_j b_j \frac{\partial f}{\partial x_j} = a_i \frac{\partial b_j}{\partial x_i} \frac{\partial f}{\partial x_j} + a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

$$YX f = Y(Xf) = b_j \frac{\partial a_i}{\partial x_j} \frac{\partial f}{\partial x_i} + a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

$$\Rightarrow (XY - YX) f = \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial f}{\partial x_j}$$

$[X, Y] = XY - YX$: Lie Bracket

Properties:

1. $[X, Y] = -[Y, X]$ ← Anti Commutative
2. $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ ← Bilinear
   $[X, aY + bZ] = a[X, Y] + b[X, Z]$
   const. scalars
3. $[[X,Y], Z] + [[Y,Z], X] + [[Z,X], Y] = 0$ ← Jacobi Identity
4. $[fX, gY] = fg[X, Y] + f X(g) Y - g Y(f) X$
   $\hookrightarrow = fX(gY) - gY(fX)$
   $= fgXY + f X(g) Y - g Y(f) X - gf YX$
   $= fg[X, Y] + f X(g) Y - g Y(f) X$

$$[X_i, X_j] = c_{ij}^{k} X_k$$

$\uparrow$ Structure Constants of the algebra

$$c_{ij}^{k} = -c_{ji}^{k}$$

$$\text{ad}_X(Y) = [X, Y]$$

$$\text{ad}(X)\, Y = [X, Y]$$

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a v.f. on $\mathbb{R}^n$.

& $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar function.

Lie Derivative of h wrt f.

$$L_f h = \frac{\partial h}{\partial x} f = \sum \frac{\partial h}{\partial x_i} f$$

$$L_f^2 h = L_f(L_f(h))$$

$$L_{[f,g]} h = L_f L_g h - L_g L_f h \quad \text{[Jacobi Identity]}$$

Proof: LHS =

$$L_{[f,g]} h = \sum \sum \frac{\partial h}{\partial x_i} \left[\frac{\partial g_i}{\partial x_j} f_j - \frac{\partial f_i}{\partial x_j} g_j \right]$$

RHS $\rightarrow$ $\quad a_i = (\partial h / \partial x_i)$

$$① \; L_f L_g h = \frac{\partial}{\partial x_j}(a_i g_i) f_j = \frac{\partial a_i}{\partial x_j} g_i f_j + a_i \frac{\partial g_i}{\partial x_j} f_j$$

$$② \; L_g L_f h = \frac{\partial a_i}{\partial x_j} f_i g_j + a_i \frac{\partial f_i}{\partial x_j} g_j$$

$\Rightarrow$ LHS = RHS

Example: $f(x) = \begin{bmatrix} x_2 \\ \sin x_1 \\ x_1 x_3^2 \end{bmatrix} \qquad g(x) = \begin{bmatrix} 0 \\ x_2^2 \\ 1 \end{bmatrix}$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 \\ \cos x_1 & 0 & 0 \\ x_3^2 & 0 & 2x_1 x_3 \end{bmatrix} \qquad \frac{\partial g}{\partial x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2x_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial x} g - \frac{\partial g}{\partial x} f = \begin{bmatrix} -x_2^2 \\ 2x_2 \sin x_1 \\ -2x_1 x_3 \end{bmatrix}$$

$$\dot{x} = f(x) + g(x) u \qquad ad_f^k(g) = [f, ad_f^{k-1}(g)] \qquad ad_f^0(g) = g$$

Example: $f(x) = \begin{bmatrix} -2x_1 + ax_2 + \sin x_1 \\ -x_2 \cos x_1 \end{bmatrix} \qquad g(x) = \begin{bmatrix} 0 \\ \cos(2x_1) \end{bmatrix}$

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

$$\frac{\partial g}{\partial x} = \begin{bmatrix} 0 & 0 \\ -2\sin 2x_1 & 0 \end{bmatrix} \qquad \frac{\partial f}{\partial x} = \begin{bmatrix} -2 + \cos x_1 & a \\ x_2 \sin x_1 & -\cos x_1 \end{bmatrix}$$

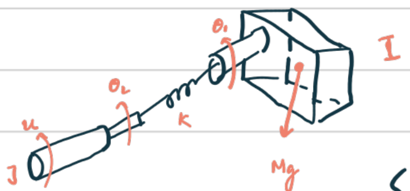
$$[f, g] = \begin{bmatrix} -a\cos 2x_1 \\ -2\sin(2x_1)(-2x_1 + ax_2 + \sin x_1) + \cos x_1 \cos 2x_1 \end{bmatrix}$$

Definition: A distribution $\Delta = \text{span}\{X_1, \cdots, X_m\}$ is said to be <u>involutive</u> iff $\exists$ scalar functions: $\alpha_{ijk} : \mathbb{R}^n \to \mathbb{R}$ s.t.

$$[X_i, X_j] = \alpha_{ijk} X_k \qquad \forall\, i, j, k$$

An involutive distribution is closed under the Lie bracket operation.

Frobenius Theorem: A distribution Δ is integrable iff it is involutive.



$$\begin{cases} I\ddot{\theta}_1 + Mg\ell \sin\theta_1 + k(\theta_1 - \theta_2) = 0 \\ J\ddot{\theta}_2 + k(\theta_2 - \theta_1) = u \end{cases}$$

$$\begin{cases} x_1 = \theta_1 & x_3 = \theta_2 \\ x_2 = \dot{\theta}_1 & x_4 = \dot{\theta}_2 \end{cases}$$

Non linear system is feedback linearizable

- Vector fields $\{g, ad_f(g), \cdots, ad_f^{n-1}(g)\}$ are lin. indep.
- Distribution $\Delta = \{g, ad_f\, g, \cdots, ad_f^{n-2}(g)\}$ is involutive

Example (contd): $f(x) = \begin{bmatrix} x_2 \\ -\frac{Mg\ell \sin x_1}{I} - \frac{K}{I}(x_1 - x_3) \\ x_4 \\ \frac{K}{J}(x_1 - x_3) \end{bmatrix}$ $\quad g(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}$

$$ad_f\, g = \begin{bmatrix} 0 \\ 0 \\ -1/J \\ 0 \end{bmatrix} \qquad \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{MgL}{I}\cos x_1 - \frac{k}{I} & 0 & k/I & 0 \\ 0 & 0 & 0 & 1 \\ k/J & 0 & -k/J & 0 \end{bmatrix}$$

$$ad_f^2\, g = \begin{bmatrix} 0 \\ k/IJ \\ 0 \\ -k/J^2 \end{bmatrix} \qquad ad_f^3\, g = \begin{bmatrix} -k/IJ \\ 0 \\ k/J^2 \\ 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 0 & 0 & 0 & -k/IJ \\ 0 & 0 & k/JI & 0 \\ 0 & -1/J & 0 & k/J^2 \\ 1/J & 0 & -k/J^2 & 0 \end{bmatrix}$$

Constraints written as $\underbrace{A(q)}_{\in \mathbb{R}^{k\times n}}\dot{q} = 0$ $\{k = \#\text{constraints}\}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q}\right) - \frac{\partial L}{\partial q} + A^T\lambda = F$$

λ ↳ Lagrange multiplier. $\in \mathbb{R}^k$

$\lambda_1, \dots, \lambda_k$

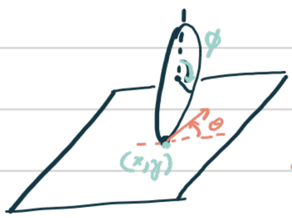
$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q,\dot{q}) + A^T(q)\lambda = F$$

Derivative of $A(q)\cdot\dot{q}$ → $\dot{A}(q)\dot{q} + A(q)\ddot{q} = 0$

$$\Rightarrow (AM^{-1}A^T)\lambda = AM^{-1}(F - C\dot{q} - N) + \dot{A}\dot{q}$$

If the constraints are independent then $AM^{-1}A^T$ is invertible:

$$\lambda = (AM^{-1}A^T)^{-1}\left[AM^{-1}(F - C\dot{q} - N) + \dot{A}\dot{q}\right]$$



$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2\dot{\phi}^2$$

$q = (x, y, \theta, \phi)$

$$\begin{cases} \dot{x}\sin\theta + \dot{y}\cos\theta = 0 \\ \dot{x} = \rho\dot{\phi}\cos\theta \\ \dot{y} = \rho\dot{\phi}\sin\theta \end{cases}$$

$$A\left\{\begin{bmatrix} 1 & 0 & 0 & -\rho\cos\theta \\ 0 & 1 & 0 & -\rho\sin\theta \end{bmatrix}\right. \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\phi}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

↖ $(\sin\theta \;\; \cos\theta \;\; 0 \;\; 0)$ dependent

$$\tau = \begin{bmatrix}0\\0\\\tau_\theta\\\tau_\phi\end{bmatrix} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + A^T\lambda = \tau$$

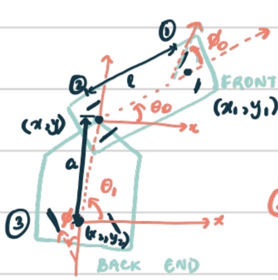
$$A^T\lambda = \begin{bmatrix}\lambda_1\\\lambda_2\\0\\-\lambda_1\rho c_\theta - \lambda_2\rho s_\theta\end{bmatrix} \qquad \begin{bmatrix} m & & & \\ & m & & \\ & & I_1 & \\ & & & I_2\end{bmatrix}\begin{bmatrix}\ddot{x}\\\ddot{y}\\\ddot{\theta}\\\ddot{\phi}\end{bmatrix} + \begin{bmatrix}\lambda_1\\\lambda_2\\0\\-\lambda_1\rho c_\theta - \lambda_2\rho s_\theta\end{bmatrix} = \begin{bmatrix}0\\0\\\tau_\theta\\\tau_\phi\end{bmatrix}$$

$$\lambda_1 = -m\ddot{x}$$
$$\lambda_2 = -m\ddot{y}$$

$$\ddot{x} = \rho c_\theta\ddot{\phi} - \rho s_\theta\dot{\phi}\dot{\theta}$$
$$\ddot{y} = \rho s_\theta\ddot{\phi} + \rho c_\theta\dot{\theta}\dot{\phi}$$

$$\begin{bmatrix} I_1 & 0 \\ 0 & I_2 + m\rho^2\end{bmatrix}\begin{bmatrix}\ddot{\theta}\\\ddot{\phi}\end{bmatrix} = \begin{bmatrix}\tau_\theta\\\tau_\phi\end{bmatrix}$$

Firetruck:



Three pair of wheels.

(2) $\dot{x}\sin\theta_0 - \dot{y}\cos\theta_0 = 0$

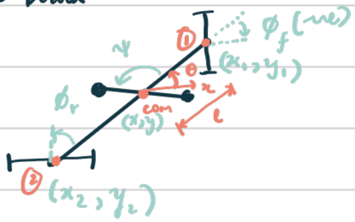
$\{x_1 = x + l\cos\theta_0, \; y_1 = y + l\sin\theta_0\}$

$$\dot{x}_1 \sin(\theta_0 + \phi_0) - \dot{y}_1 \cos(\theta_0 + \phi_0) = 0$$

$$① \quad \dot{x} \sin(\theta_0 + \phi_0) - \dot{y} \cos(\theta_0 + \phi_0) - l\,\dot{\theta}_0 \cos\phi_0 = 0$$

$$③ \quad \dot{x} \sin(\phi_1 + \theta_1) - \dot{y} \cos(\phi_1 + \theta_1) + a\,\dot{\theta}_1 \cos\phi_1 = 0$$

Snake board



m: mass of the system

l: half length

J_r: Moment of inertia of control rod

J_w: " of wheels

$$\left.\begin{array}{l}\tau_\psi \\ \tau_{\phi_f} \\ \tau_{\phi_r}\end{array}\right\} \leftarrow \text{Input torques}$$

of center rod

$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \left\{\frac{1}{2} J (\dot{\theta}^2)\right\} + \frac{1}{2} J_w (\dot{\theta} + \dot{\phi}_f)^2 + \frac{1}{2} J_w (\dot{\theta} + \dot{\phi}_r)^2 + \left\{\frac{1}{2} J_r (\dot{\theta} + \dot{\psi})^2\right\}$$

Constraints: ② $\dot{y} \cos(\phi_r + \theta) - \dot{x} \sin(\phi_r + \theta) - l\dot{\theta} \cos\phi_r = 0$

① $\dot{y} \cos(\phi_f + \theta) - \dot{x} \sin(\phi_f + \theta) + l\dot{\theta} \cos\phi_f = 0$

$$A(q) = \begin{bmatrix} \sin(\phi_r + \theta) & -\cos(\phi_r + \theta) & l\cos\phi_r & 0 & 0 & 0 \\ \sin(\phi_f + \theta) & -\cos(\phi_f + \theta) & -l\cos\phi_f & 0 & 0 & 0 \end{bmatrix}$$

$(x, y, \theta, \psi, \phi_r, \phi_f)$

(Understanding internal forces & external)

Chp 4 - Murray Slotine
Robot analysis & Control

$$m\ddot{x} + \lambda_1 \sin(\phi_r + \theta) + \lambda_2 \sin(\phi_f + \theta) = 0$$

$$m\ddot{y} - \lambda_1 \cos(\phi_r + \theta) - \lambda_2 \cos(\phi_f + \theta) = 0$$

Equations of motion

$$\left\{\begin{array}{l}(J + J_r + 2J_w)\ddot{\theta} + J_r\ddot{\psi} + J_w\ddot{\phi}_r + J_w\ddot{\phi}_f + \lambda_1 l \cos\phi_r - \lambda_2 l \cos\phi_f = 0 \\ J_r\ddot{\psi} + J_r\ddot{\theta} = \tau_\psi \\ J_w\ddot{\phi}_r + J_w\ddot{\theta} = \tau_{\phi_r} \\ J_w\ddot{\phi}_f + J_w\ddot{\theta} = \tau_{\phi_f}\end{array}\right.$$

Example 4.3

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \xi_2 = \begin{bmatrix} 0 \\ -l_0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \xi_3 = \begin{bmatrix} 0 \\ -l_0 \\ \phi_1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$g_{sl_1}(0) = \left[\begin{array}{c|c} I & \begin{matrix} 0 \\ 0 \\ r_0 \end{matrix} \\ \hline 0 & 1 \end{array}\right] \qquad g_{sl_2}(0) = \left[\begin{array}{c|c} I & \begin{matrix} 0 \\ r_1 \\ l_0 \end{matrix} \\ \hline 0 & 1 \end{array}\right]$$

$$g_{sl_3}(0) = \left[\begin{array}{c|c} I & \begin{matrix} 0 \\ l_1+r_2 \\ l_0 \end{matrix} \\ \hline 0 & 1 \end{array}\right]$$

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix}, \quad q_3 = \begin{bmatrix} 0 \\ l_1 \\ l_0 \end{bmatrix}, \quad \omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \omega_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \omega_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$J_1 = J^b_{sl_1}(0) = \begin{bmatrix} \xi_1^\dagger & 0 & 0 \end{bmatrix}$$

$$\xi_j^\dagger = Ad^{-1}_{\left(e^{\hat{\xi}_j \theta_j} \cdots e^{\hat{\xi}_i \theta_i} g_{sl_i}(0)\right)} \xi_i$$

$$\xi_1^\dagger = Ad^{-1}_{\left(e^{\hat{\xi}_1 \theta_1} g_{sl_1}(0)\right)} \xi_1$$

$$g_1' = e^{\hat{\xi}_1 \theta_1} g_{sl_1}(0) = \left[\begin{array}{ccc|c} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & r_0 \\ \hline & 0 & & 1 \end{array}\right]$$

$$Ad^{-1}_{g_1} \xi_1 = \left[\begin{array}{c|c} R & \begin{matrix} -s_1 r_0 & c_1 r_0 & 0 \\ -c_1 r_0 & -s_1 r_0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \hline 0 & R \end{array}\right] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad J_1 = \begin{bmatrix} 0 & & \\ 0 & & \\ 0 & & \\ 0 & 0 & 0 \\ 0 & & \\ 1 & & \end{bmatrix}$$

$$J_2 = \begin{bmatrix} \xi_1^\dagger & \xi_2^\dagger & 0 \end{bmatrix}$$

$$\xi_1^\dagger = Ad^{-1}_{\left(\underbrace{e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} g_{sl_2}(0)}_{g_2'}\right)} \xi_1$$

$$\xi_2^\dagger = Ad^{-1}_{\left(e^{\hat{\xi}_2 \theta_2} g_{sl_2}(0)\right)} \xi_2$$

$$e^{\hat{\xi}_1 \theta_1} = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -l_0 s_2 \\ (1-c_2) l_0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$e^{\hat{\xi}_2 \theta_2} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & c_2 & s_2 & \left((I - e^{\hat{\omega}_2 \theta_2})(\omega_2 \times v_2)\right) + \omega_2 \omega_2^T v_2 \theta_2 \\ 0 & -s_2 & c_2 & \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Not from frame 1 to frame 2

$$g_2' = \left[\begin{array}{ccc|c} c_1 & -s_1 c_2 & -s_1 s_2 & -r_1 s_1 c_2 \\ s_1 & c_1 c_2 & c_1 s_2 & r_1 c_1 c_2 \\ 0 & -s_2 & c_2 & l_0 - r_1 s_2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \qquad \xi_1^\dagger = \begin{bmatrix} -r_1 c_2 \\ 0 \\ 0 \\ 0 \\ -s_2 \\ c_2 \end{bmatrix} \quad \xi_2^\dagger = \begin{bmatrix} 0 \\ 0 \\ -r_1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$J_3 = J^b_{g_{l_3}(0)} = \begin{bmatrix} \xi_1^\dagger & \xi_2^\dagger & \xi_3^\dagger \end{bmatrix}$$

$$\xi_1^{\dagger} = Ad^{-1}\left(e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} g_{sl_3}(0)\right)\xi_1 = Ad^{-1}_{g_3^1}\xi_1$$

$$\xi_2^{\dagger} = Ad^{-1}\left(e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} g_{sl_3}(0)\right)\xi_2 = Ad^{-1}_{g_3^2}\xi_2$$

$$\xi_3^{\dagger} = Ad^{-1}\left(e^{\hat{\xi}_3\theta_3} g_{sl_3}(0)\right)\xi_3 = Ad^{-1}_{g_3^3}\xi_3$$

$$J_3 = \begin{bmatrix} -l_2c_2 - r_2c_{23} & 0 & 0 \\ 0 & l_1 s_3 & 0 \\ 0 & -r_2 - l_1c_3 & -r_2 \\ 0 & -1 & -1 \\ -s_{23} & 0 & 0 \\ c_{23} & 0 & 0 \end{bmatrix}$$

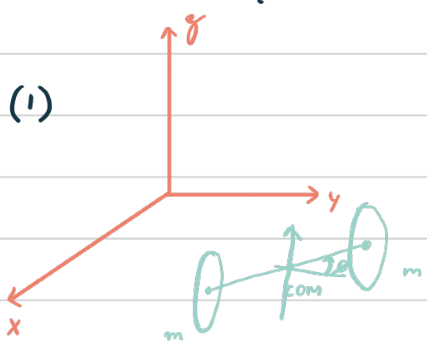
$$M(\theta) = J_1 \mu_1 J_1 + \underbrace{J_2^T \mu_2 J_2}_{} + J_3^T M_3 J_3$$

$$\mu_2 = \begin{bmatrix} m_2 & & & & & \\ & m_2 & & & 0 & \\ & & m_2 & & & \\ & & & I_{x_2} & & \\ & 0 & & & I_{y_2} & \\ & & & & & I_{z_2} \end{bmatrix}$$

$$J_2^T M_2 J_2 = \begin{bmatrix} m_2 r_1^2 c_2^2 + I_{y_2} s_2^2 + I_{z_2} c_2^2 & 0 & 0 \\ 0 & m_2 r_1^2 + I_{x_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problems

(1)



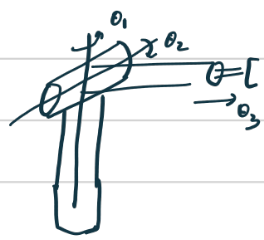
$\phi(0) = 0$

$\dot{\phi}(0) = \omega$

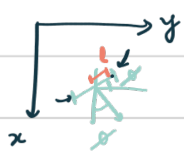
com is moving with velocity v_0

Find position of com as a function of time.

(2) Find spatial Jacobian
3link manipulator



(1)



State variables : (x, y, ϕ)

Constraint : $\dot{x}\sin\phi - \dot{y}\cos\phi = 0 \rightarrow A = \begin{bmatrix} \sin\phi & -\cos\phi & 0 \end{bmatrix}$

$T = 0$

1^{st} disk center : $(x - l\sin\phi,\ y + l\cos\phi)$

2^{nd} disk center : $(x + l\sin\phi,\ y - l\cos\phi)$

$$V = \frac{1}{2} m \Big((\dot{x} - l \cos\phi \, \dot{\phi})^2 + (\dot{y} - l \sin\phi \, \dot{\phi})^2 + (\dot{x} + l \cos\phi \, \dot{\phi})^2 + (\dot{y} + l \sin\phi \, \dot{\phi})^2 \Big)$$

$$= \frac{1}{2} m \left(2\dot{x}^2 + 2\dot{y}^2 + l^2 \dot{\phi}^2 \right)$$

$$\mathcal{L} = - m\dot{x}^2 - m\dot{y}^2 - \frac{1}{2} m l^2 \dot{\phi}^2$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} + A^T \lambda = F$$

x: $\quad -2m\ddot{x} + \lambda \sin\phi = 0$

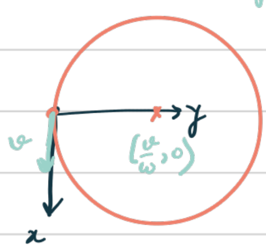
y: $\quad -2m\ddot{y} - \lambda \cos\phi = 0$

ϕ: $\quad \ddot{\phi} = 0 \quad \Rightarrow \quad \dot{\phi} = \text{const.}$

$\Rightarrow \quad \phi(t) = \omega t$

$\left(+2m\, v\omega \sin\omega t + \lambda \sin\omega t = 0 \quad \Rightarrow \quad \boxed{\lambda = -2mv\omega} \right)$

$\dot{x}(0) = v$, $\dot{y}(0) = 0$ $\quad \Rightarrow \quad$ $\dot{x}(t) = v \cos(\omega t)$, $\dot{y}(t) = v \sin(\omega t)$ $\quad \Rightarrow \quad$ $x(t) = \frac{v}{\omega} \sin \omega t$, $y(t) = \frac{v}{\omega}(1 - \cos\omega t)$



$$\boxed{x^2 + \left(y - \frac{v}{\omega}\right)^2 = \left(\frac{v^2}{\omega^2}\right)}$$

Homework Review Notes

Group Homomorphism: $\phi(a \cdot b) = \phi(a) * \phi(b)$

Subgroup: closure, identity & inverse

Normal Subgroup: $N \subset G$, $\forall x \in G$ if $xNx^{-1} = N$

Cayley Parameterization: $\{A = -A^T\}$ $R = (I+A)(I-A)^{-1} \in SO(3)$

$$V^b_{ac} = Ad_{g^{-1}_{bc}} V^b_{ab} + V^b_{bc}$$

$$g_{ac} = g_{ab}\, g_{bc}$$

$$\hat{V}^b_{ac} = g^{-1}_{ac}\, \dot{g}_{ac}$$

$$g = \left[\begin{array}{c|c} R & p \\ \hline 0 & 1 \end{array}\right] \qquad g^{-1} = \left[\begin{array}{c|c} R^T & -R^T p \\ \hline 0 & 1 \end{array}\right]$$

$$Ad_g = \left[\begin{array}{c|c} R & DR \\ \hline 0 & R \end{array}\right] \quad \{D = \hat{p}\}$$

$$Ad_{g^{-1}} = \left[\begin{array}{c|c} R^T & -R^T D \\ \hline 0 & R^T \end{array}\right]$$